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STABILITY STUDIES OF PERIODIC SOLUTIONS

FINAL REPORT

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FOREWORD

The principal objective of Contract NAS8-20323, "Research in Stability of Periodic Motions," is to derive exact analytical results concerning the degree of instability of certain periodic orbits in the restricted three-body problem. The specialized nature of this problem required the assimilation of background material before specific aspects can be studied. This report covers such material.

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SUMMARY

The stability of periodic orbits of the restricted three-body problem is discussed. Available techniques are presented and the necessary background material is developed.

The existence of periodic solutions of the restricted three-body problem is developed from first principles. Starting with Newton's gravitational law and equations of motion the proof is developed via the Lagrange and Hamilton equations of motion, existence theorem for differential equations, the two and three-body problems.

CONTENTS

Section 1
INTRODUCTION

Before a periodic orbit (e.g., one enclosing the Earth and Moon) becomes useful for space exploration, we must know something of its stability characteristics. To be useful, a periodic orbit need not be stable but rather deviate slowly from the initial or intended orbit. In this case, we talk about the lifetime of an orbit. This lifetime may be increased significantly by "correcting" the orbit periodically, providing sufficient energy is available. The intent herein is to familiarize one with the techniques available for determining the stability characteristics of such orbits and how they may be applied.

Section 2
TECHNICAL DISCUSSION

2.1 BASIC MECHANICS

The first step in establishing a mathematical approach to celestial mechanics is the acceptance of Newton's laws. These laws, based upon observations made during the late sixteenth century and early seventeenth, have proven to be only a first approximation to Einstein's relativistic laws, but in fact the inadequacies of the system, so far as celestial mechanics is concerned, arise as much from the difficulty of the measurement of distance and time as from the relativistic connection between the pair.

We begin, then, by assuming the concept of time, and take as its measurement what is known as ephemeris time rather than universal time and define absolute space as any space in which Newton's laws hold. To be rigorous, then, we have divorced ourselves from the real world, even though we do not know just what is the real world, but, if we note that the advance of the perihelion of Mercury is some 43 seconds of arc per century, we do not feel that we are so far removed from reality that our approach is useless.

Now briefly, Newton's second law states that for constant mass, the product of the mass and the acceleration is equal to the applied external force. This gives the familiar equation

$$m_i \ddot{r}_i = F_i \qquad (i = 1, \ldots, n) \tag{1}$$

where n is the number of particles in the system.

His gravitational law allows us to compute F_i. It states that the force between any one body of a system and a second body is directly proportional to the product of their masses and inversely proportional to the square of the distance between them, acts along the straight line joining their centers and is independent of any other forces acting. This allows us to write that the force, F_{ij}, acting on the i^{th} particle due to the j particle is just

$$F_{ij} = -\gamma\, m_i m_j \frac{r_{ij}}{|r_{ij}|^3} \,. \tag{2}$$

where r_{ij} is the position vector of m_j relative to m_i (i.e., $r_{ij} = r_j - r_i$), r_i, r_j being the position vectors of m_i and m_j, respectively, relative to some arbitrary origin. The total gravitational force on the i^{th} particle is then

$$F_i = -\gamma m_i \sum_{j \neq i} m_j \frac{r_j - r_i}{|r_j - r_i|^3} \,, \tag{3}$$

and Newton's equations of motion become

$$m_i \ddot{r}_i = -\gamma\, m_i \sum_{j \neq i} m_j \frac{r_j - r_i}{|r_j - r_i|^3} \,, \qquad (i, j = 1, \ldots, n). \tag{4}$$

Now we may show by differentiation that the force F_i is a gradient of a scalar function. By way of definition, we write

$$F_i = -\mathrm{grad}_{r_i} V \tag{5}$$

where $\mathrm{grad}_{r_i} V = (V_{x_i}, V_{y_i}, V_{z_i})$. V is called the potential function or potential energy, and we see that

$$V = -\gamma \frac{1}{2} \sum_{i} \sum_{j \neq i} m_j \frac{1}{|r_j - r_i|} \cdot \tag{6}$$

We define the kinetic energy T to be

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{r}_i^2 \ . \tag{7}$$

The ten classical integrals give us the preservation of linear momentum, angular momentum and total energy (T + V). The first of these allows us to take the origin at the center of mass of the system without changing the form of the equations, which we shall do.

Before continuing with the derivation of the two- and three-body problem from the general n body problem, we shall discuss the Lagrange and Hamilton formulations as equivalents to Newton's equations of motion. The power and usefulness of these formulations will be demonstrated in later sections.

We define the Lagrangian

$$L = T - V \tag{8}$$

and see that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}_i}\right) = m_i \ddot{\underline{x}}_i$$

and

$$\frac{\partial L}{\partial \underline{x}_i} = -\frac{\partial V}{\partial \underline{x}_i} = -\gamma \ m_i \sum_{j \neq i} m_j \frac{\underline{x}_i - \underline{x}_i}{|\underline{x}_j - \underline{x}_i|^3}$$

for $\underline{x}_i = x_i, y_i, z_i$ and $i, j = 1, \ldots, n$.

We have then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{x}}_i}\right) - \frac{\partial L}{\partial \underline{x}_i} = 0. \tag{9}$$

To make this more general, we consider a non-singular transformation of the x_i, y_i, z_i $(i = 1, \ldots n)$ to what we call generalized coordinates q_j. Now, we have 3n cartesian coordinates, so we require 3n generalized coordinates q_j, provided no constraints are placed upon the motion, for the transformation to be non-singular. Let us consider a transformation of the form

$$\underline{x}_i = \underline{x}_i(q_1, \ldots, q_{3n}), \quad (i = 1, \ldots n) \tag{10}$$

then

$$L = L^*(q_1, \ldots, q_{3n}).$$

We shall prove the form of the equations is invariant by direct evaluation. The first term of Equation (1) is

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{\underline{x}}_i}\frac{\partial \dot{\underline{x}}_i^*}{\partial \dot{q}_j} + \sum_i \frac{\partial L}{\partial \underline{x}_i}\frac{\partial \underline{x}_i^*}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{\underline{x}}_i}\frac{\partial \dot{\underline{x}}_i^*}{\partial \dot{q}_j}\right)$$

since $\underline{x}_i^*$ is independent of the $\dot{q}_s$, and

$$\left(\frac{\partial L^*}{\partial q_j}\right) = \sum_i \frac{\partial L}{\partial \underline{x}_i}\frac{\partial \underline{x}_i^*}{\partial q_j} + \sum_i \frac{\partial L}{\partial \dot{\underline{x}}_i}\frac{\partial \dot{\underline{x}}_i^*}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_j}\right) - \frac{\partial L^*}{\partial q_j} = \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\frac{\partial \dot{x}_i^*}{\partial q_i} + \sum \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}\left(\frac{\partial \dot{x}_i^*}{\partial \dot{q}_j}\right)$$

$$- \sum \frac{\partial L}{\partial x_i}\frac{\partial x_i^*}{\partial q_i} - \sum \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i^*}{\partial q_j} \tag{11}$$

The 1st and 3rd terms of Equation (11) combine to give zero on using Equation (9) on the following fact:

$$\frac{\partial \dot{x}_i^*}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{dx_i^*}{dt} = \frac{\partial}{\partial \dot{q}_j}\sum_\alpha \frac{\partial x_i^*}{\partial q_\alpha}\dot{q}_\alpha = \sum_\alpha \frac{\partial x_i^*}{\partial q_\alpha}\frac{\partial \dot{q}_\alpha}{\partial \dot{q}_j} = \frac{\partial x_i^*}{\partial q_j}$$

i.e.,

$$\frac{\partial \dot{x}_i^*}{\partial \dot{q}_j} = \frac{\partial x_i^*}{\partial q_j}$$

Also using the above fact and a reversal of the order of differentiation allows a cancellation of the 2nd and 4th terms. Substituting these into Equation (11) gives

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 , \tag{12}$$

i.e., the form of the equations is invariant. We notice, in general, that the system is not first order, for L contains a term $\dot{q}_\alpha^2$ and so $\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_\alpha}\right)$ contains a term $\ddot{q}_\alpha$.

On introducing the Lagrangian derivative

$$\mathcal{L}_{x_j} \equiv \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_j}\right) - \frac{\partial}{\partial x_j} ,$$

we may write Equation (12) as

$$\mathcal{L}_{x_j}(L) = 0$$

The next step is to introduce the Hamiltonian system. We define the canonical momenta p_α by

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \qquad (\alpha = 1, \ldots, n) \tag{13}$$

Now we put

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L \tag{14}$$

where we consider H a function of p_α, q_α ($\alpha = 1, \ldots, n$). This is accomplished on solving Equation (13) for $\dot{q}_\alpha$, and substituting in Equation (14). We denote L as a function of p, q by L^*.

Now

$$H_{q_\alpha} = \sum_\beta p_\beta \frac{\partial \dot{q}_\alpha}{\partial q_\alpha} - \sum_\beta \frac{\partial L}{\partial \dot{q}_\beta} \frac{\partial \dot{q}_\beta}{\partial q_\alpha} - \frac{\partial L}{\partial q_\alpha} \quad .$$

The first and second terms combine to give zero, and the third term is just $-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}$ by Equation (9). But $\frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha$ and so

$$H_{q_\alpha} = -\dot{p}_\alpha \qquad (\alpha = 1, \ldots, n) \; .$$

Now we consider

$$H_{p_\alpha} = q_\alpha - \frac{\partial L^*}{\partial p_\alpha} + \sum_\beta p_\beta \frac{\partial q_\beta}{\partial p_\alpha} \quad .$$

Now $\frac{\partial L^*}{\partial p_\alpha} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial p_\alpha}$, $\left(\text{no second term, since } \frac{\partial q_\beta}{\partial p_\beta} = 0\right)$

$$\therefore H_{p_\alpha} = \dot{q}_\alpha \quad .$$

We have then, the 2n equations

$$\begin{aligned} \dot{q}_\alpha &= H_{p_\alpha} \\ \dot{p}_\alpha &= -H_{q_\alpha} \end{aligned} \qquad (\alpha = 1, \ldots, n) \tag{15}$$

to describe our system. These equations we call Hamilton's equations of motion, or sometimes, the canonical equations of motion. Providing we restrict ourselves to nonsingular transformations throughout, we can return via the inverse transformations to our original system of Newtonian equations.

Invariance of the Hamiltonian Equations

We define a canonical transformation of the variables p_α, q_α $(\alpha = 1, \ldots, n)$ as one which preserves the form of the Hamilton equations. We now rename our variables, which is in reality an identity transformation, and consider a system of the Hamiltonian equations

$$\begin{aligned} \dot{x}_i &= H_{y_i} \\ \dot{y}_i &= -H_{x_i} \end{aligned} \qquad (i = 1, \ldots, n) . \tag{16}$$

We define the column vector z with $2n$ elements z_i $(i = 1, \ldots, 2n)$ by $z_j = x_j$, $z_{j+n} = y_n$ $(j=1, \ldots, n)$. Formally then, the transformation from z to ζ with $\zeta_i = \xi_i$ and $\zeta_{i+n} = \eta_i$ $(i = 1, \ldots, n)$ may be written

$$z = z^* (\zeta, t) . \tag{17}$$

In order to ensure the existance of the inverse transformation, we require the matrix of partials

$$M = z_\zeta = (z_{i\zeta_j}) = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \tag{18}$$

to be nonsingular. Our last preparatory step before transforming the equations is to introduce the $2n \times 2n$ matrix J,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the n x n identity matrix. We notice that

$$J^{-1} = -J = J^T .$$

We may now write our equations of motion as

$$\dot{z} = J H_z \tag{19}$$

where H_z is the column vector with 2n elements $H_{z_i} = H_{x_i}$, $H_{z_{i+n}} = H_{y_i}$ $(i = 1, \ldots, n)$.

Now

$$\dot{z}_i = \sum_{i=1}^{2n} \frac{\partial z_i^*}{\partial \zeta_j} \dot{\zeta}_j + \frac{\partial z^*}{\partial t} \qquad (i = 1, \ldots, 2n)$$

where the * quantity denotes that it is considered a function of (ζ, t). But the first term on the right hand side is just the i^{th} term of the column vector $M^{-1}\dot{\zeta}$. We may write, then

$$\dot{z} = M^{-1}\dot{\zeta} . \tag{20}$$

The general term H_{z_i} must now be transformed. We have

$$H_{z_i} = \frac{\partial H}{\partial z_i} = \sum_j \frac{\partial H^*}{\partial \zeta_j} \frac{\partial \zeta_j}{\partial z_i} .$$

But $\sum_j \frac{\partial \zeta_j}{\partial z_i} \frac{\partial H^*}{\partial \zeta_j}$ is the i^{th} element of the column vector $\left(\frac{\partial \zeta_j}{\partial z_i}\right)^T H_\zeta^*$, and ζ_z is just $(z_\zeta)^{-1} = M^{-1}$ and so Equation (19) becomes

$$M^{-1}\dot{\zeta} = J\,M^T H^* - \frac{\partial z^*}{\partial t}$$
$$\therefore \dot{\zeta} = M\,J\,M^T H^* - \frac{\partial z^*}{\partial t} \tag{21}$$

To preserve the form of the equations, we must be able to write in the form

$$\dot{\zeta} = J\,\widetilde{H}_{\zeta}\,. \tag{22}$$

We now equate the right hand sides of Equations (21) and (22) and multiply through by $-J$ to give

$$\widetilde{H}_{\zeta} = -JMJM^T H^*_{\zeta} + J z^*_t\,.$$

Clearly, if $MJM^T = J$ and $-Jz^*_t$ is a gradient, i.e., $-J z^*_t$ can be written as R_{ζ}, then we have

$$\widetilde{H} = H^* - R\,.$$

If our transformation is conservative, then $z^*_t = 0$ and

$$\widetilde{H} = H^*\,. \tag{23}$$

We have shown above that $M^{-1} J M^{-1^T} = J$ and $-J z^*_t$ a gradient is a sufficient condition for a transformation to be canonical. Before proceeding, we should note that in the literature, we frequently find that a canonical transformation is defined somewhat more generally by

$$M\,J\,M^T = \mu\,J \tag{24}$$

where μ is a constant different from zero, and the case, $\mu = 1$ is called symplectic.

Now that we have this result, we prove the following interesting result, namely, that every function $v = v(x, \eta)$ generates a simplectic transformation, $y = v_x$, $\xi = v_\eta$, provided $\det(v_{\eta x}) \neq 0$.

When we say this generates a transformation, we mean that we solve $\xi = v_\eta$ for x, hence the condition $\det(v_{\eta x}) \neq 0$, and write

$$x = \varphi(\xi, \eta) \tag{25}$$

and substitute for x in $y = v_x$, to obtain

$$y = \psi(\xi, \eta) . \tag{26}$$

Now to prove this result, we note that

$$M^T J M = \begin{pmatrix} \varphi_\xi & \varphi_\eta \\ \psi_\xi & \psi_\eta \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \varphi_\xi & \varphi_\eta \\ \psi_\xi & \psi_\eta \end{pmatrix}$$

$$= \begin{pmatrix} -(\psi_\xi)^T (\varphi_\eta) + (\varphi_\eta)^T (\psi_\xi) & -(\psi_\xi)^T (\varphi_\eta) + (\varphi_\xi)^T (\psi_\eta) \\ -(\psi_\eta)^T (\varphi_\xi) + (\varphi_\eta)^T (\psi_\xi) & -(\psi_\eta)^T (\varphi_\eta) + (\varphi_\eta)^T (\psi_\eta) \end{pmatrix} . \tag{27}$$

We must prove that the final matrix in Equation (27) is just J.

From Equation (26), we have, on taking the partial derivative with respect to ξ_k

$$\psi_{k\xi_j} = \sum_{r=1}^{n} v_{x_k x_r} \varphi_{r\xi_j} \qquad (j, k = 1, \ldots, n)$$

which is just the element in the k^{th} column of j^{th} row of $k_{xx}\varphi_\xi$, and so

$$\psi_\xi = v_{xx} \varphi_\xi \quad . \tag{28}$$

Next consider

$$\xi_{k\xi_j} = \sum_{r=1}^{n} v_{\eta_k x_r} \varphi_{r\xi} \qquad (j, k = 1, \ldots, n)$$

which is the k, j element of

$$I = \xi_\xi = v_{\eta x} \varphi_\xi \tag{29}$$

where I is the unit matrix of order n. This gives us φ_ξ and, on using Equation (28), ψ_ξ. Now, our new coordinates are ξ, η and are independent, therefore, using arguments similar to the above

$$0 = \xi_\eta = v_{\eta x} \varphi_\eta + v_{\eta \xi}, \tag{30}$$

and, on considering ψ_k $(k = 1, \ldots, n)$, we derive

$$\psi_\eta = v_{xx} \varphi_\eta + v_{x\eta}. \tag{31}$$

Using Equations (28) through (31) to evaluate the final matrix of Equation (27) gives

$$-(\psi_\xi)^T (\varphi_\xi) + (\varphi_\xi)^T (\psi_\xi) = -(v_{xx} \varphi_\xi)^T (\varphi_\xi) + (\varphi_\xi)^T v_{xx} \varphi_\xi = 0$$

since v_{xx} is symmetric,

$$-(\psi_\xi)^T (\varphi_\eta) + (\varphi_\xi)^T (\psi_\eta) = -\varphi_\xi v_{xx} \varphi_\eta + v_{\eta x}^{-1\,T} v_{xx} \varphi_\eta + v_{\eta x}^{-1\,T} v_{x\eta} = I$$

since $v_{x\eta}$ is symmetric,

$$-(\psi_\eta)^T (\varphi_\eta) + (\varphi_\eta)^T (\psi_\eta) = -(v_{xx} \varphi_\eta)^T \varphi_\eta - v_{x\eta}^T \varphi_\eta + (\varphi_\eta)^T v_{xx} \varphi_\eta$$
$$+ \varphi_\eta^T v_{xy} = 0 .$$

Now $\xi_k = v_{\eta_k} (\varphi, \eta, t)$, but this has to be true for all systems of equations and at all points, so the partial derivative of ξ_k with respect to time is zero, i.e.,

$$0 = \sum v_{\eta_k x_r} \varphi_{r_t} + v_{\eta_k t} \qquad (k = 1, \ldots, n) .$$

But this is just the k^{th} term of the column vector $v_{\eta x} \varphi_t + v_{\eta t} = 0$. A little matrix algebra and use of the condition $\det(v_{\eta x}) \neq 0$ gives

$$\varphi_t = -v_{\eta x}^{-1} v_{\eta t}.$$

Now

$$\psi_k = v_{x_k}(\varphi, \eta, t) \tag{32}$$

but ψ_k is a function of x, η, t, and so we must write

$$\psi_{kt} = \sum_r v_{x_k x_r} \varphi_{rt} + v_{x_k t} \qquad (k = 1, \ldots, n)$$

and this gives

$$\psi_t = v_{xx}\varphi_t + v_{xt}. \tag{33}$$

A brief review is probably in order here so that we realize clearly what we are attempting. We have a Hamiltonian system

$$\dot{z} = J H_z ,$$

which we are transforming to $\dot{\zeta} = J G_\zeta$ by the transformation

$$y_k = v_{x_k}, \quad \xi_k = v_{\eta_k}, \quad v = v(x, \eta, t), \quad \det(v_{\eta x}) \neq 0 ,$$

which implies

$$x = \varphi(\xi, \eta, t), \quad y = \psi(\xi, \eta, t) .$$

By direct substitution, we have $\dot{\zeta} = J \widetilde{H}_\zeta - M^T J z_t$. We wish to show that $v_{t\zeta}(\varphi, \eta, t) = M^T J z_t$.

We have at our disposal

$$\varphi_t = -v_{\eta x}^{-1} v_{\eta t}, \quad \psi_t = v_{xx}\varphi_t + v_{xt},$$

which gives

$$z_t = \left(\begin{array}{c|c} -v_{\eta x}^{-1} & v_{\eta t} \\ \hline v_{xt} & -v_{xx} \quad v_{\eta x}^{-1} \quad v_{\eta t} \end{array}\right).$$

Next, we see that

$$v_{t\xi_j} = \sum_{r=1}^{n} v_{tx_r} \varphi_{r\xi_j}$$

$$v_{t\eta_j} = \sum_{r=1}^{n} v_{tx_r} \varphi_{r\eta_j} + h_{t\eta_j}$$

so that

$$v_{t\zeta} = \begin{pmatrix} v_{t\xi} \\ v_{t\eta} \end{pmatrix} = \begin{pmatrix} \varphi_\xi \; v_{tx} \\ \varphi_\eta^T \; v_{tx} + v_{t\eta} \end{pmatrix}$$

$$M^T J = \begin{pmatrix} \varphi_\xi^T & \psi_\xi^T \\ \varphi_\eta^T & \psi_\eta^T \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} -\psi_\xi^T & \varphi_\xi^T \\ -\psi_\eta^T & \varphi_\eta^T \end{pmatrix}$$

$$M^T J z_t = \begin{pmatrix} -\psi_\xi^T & \varphi_\xi^T \\ -\psi_\eta^T & \varphi_\eta^T \end{pmatrix} \begin{pmatrix} -v_{\eta x}^{-1} v_{\eta t} \\ v_{xt} - v_{xx} v_{\eta x}^{-1} v_{\eta t} \end{pmatrix}$$

$$= \begin{pmatrix} \psi_\xi^T v_{\eta x}^{-1} v_{\eta t} + \varphi_\xi^T v_{xt} - \varphi_\xi^T v_{xx} v_{\eta x}^{-1} v_{\eta t} \\ \psi_\eta^T v_{\eta x}^{-1} v_{\eta t} + \varphi_\eta^T v_{xt} - \varphi_\eta^T v_{xx} v_{\eta x}^{-1} v_{\eta t} \end{pmatrix} = \begin{pmatrix} \varphi_\xi^T v_{xt} \\ \varphi_\eta^T v_{tx} + v_{t\eta} \end{pmatrix}$$

$$= v_{t\zeta}$$

since $\varphi_\xi^T v_{xx} = (v_{xx} \varphi_\xi)^T = \frac{\partial}{\partial \xi}\left(v_x(\varphi, \eta, t)\right)^T = \psi_\xi^T$ and from Equation (31),

$$\psi_\eta^T v_{\eta x}^{-1} v_{\eta t} = (v_{xx} \varphi_\eta + v_{x\eta})^T v_{\eta x}^{-1} v_{\eta t}$$

$$= \varphi_\eta^T v_{xx} v_{\eta x}^{-1} v_{\eta t} + v_{x\eta}^T v_{\eta x}^{-1} v_{\eta t} .$$

We have also used that $v_{x\eta} = v_{\eta x}$, a symmetric matrix. Our final result, then, is that

$$y_k = v_{x_k}(x, \eta, t), \quad \xi_k = v_{\eta_k}(x, \eta, t) \qquad (k = 1, \dots, n)$$

defines a canonical (symplectic, if we allow $\mu \neq 1$) transformation, such that the new Hamiltonian is

$$G = H\Big(\varphi(\xi, \eta, t), \psi(\xi, \eta, t), t\Big) + H_t(\varphi, \psi, t).$$

This method of generating a canonical transformation will be used later in the normalization of the Hamiltonian. We notice that $v(x, \eta, t)$ is arbitrary, provided $v_{\eta x}$ exists and is nonsingular. In particular, we may express $v(x, \eta, t)$ as a convergent power series in x, η. In our later studies, we shall generate the power series termwise in the hope that is is convergent.

Thic concludes our discu ssion of fundamentals of mechanics.

2.2 DIFFERENTIAL EQUATIONS

We now wish to prove an existing theorem for differential equations. Not only will we prove the existance of a solution, but give a construction of the solution which is readily adopted to numerical computation.

Let $y = (y_1, \dots, y_n)$, $y \epsilon C^n$ and let $f(t, y)$ be a vector with elements functions of t, and again $f(t, y) \epsilon C^n$. We define the norm of a vector y as

$$||y|| = \max_{1 \leq j \leq n} |y_i|$$

a definition that satisfies the four requirements

$$||x|| \geq 0, \quad ||x|| = 0 \Leftrightarrow x = 0, \quad ||x + y|| \leq ||x|| + ||y|| \quad \text{and} \quad \xi \epsilon \mathscr{C} \Rightarrow ||\xi y|| = |\xi| \, ||y||,$$

where C^n denotes the set of all continuous functions with continuous derivatives up to and including the n^{th}.

By the differential equation

$$\dot{y} = f(t, y) \tag{34}$$

we mean the n first-order differential equations

$$\dot{y}_j = f_j(t, y_1, \ldots, y_n) \quad , \quad (j = 1, \ldots, n) .$$

With the preceeding definitions, we may now state our theorem:

Let f(t, y) be continuous and uniformly Lipschitz-continuous with respect to y in B: $t \subset \zeta$, $\|y - y_o\| \leq b$, where ζ is a rectifiable curve of Length L, and $\|f(t, y)\| \leq M$ on B and $ML \leq b$. Then $\dot{y} = f(b, y)$ has a unique solution $y = y(t)$ on ζ with $y(t_o) = y_o$.

Before proceeding with the proof, we consider briefly the implications of some of the requirements. The Lipschitz condition with respect to y, or as we have called it, uniformly Lipschitz continuous, it that

$$\|f(t, y) - f(t, y^*)\| \leq K \|y - y^*\| \tag{35}$$

for $t, y, y^*, \subset B$, where K is a constant independent of t, y, y^*. This means that for all $t \subset \zeta, y, y^* \subset B_o \equiv \|y - y_o\| \leq b$ (we call B_o a ball center y_o radius b, and by $\equiv$ we mean a defining equation)

$$\frac{\|f(t, y) - f(t, y^*)\|}{\|y - y^*\|}$$

is bounded, $y \neq y^*$ of course. The Lipschitz condition is satisfied if $f_y \equiv (f_{k\,y\ell})$ exists, in which case we may write

$$\|f(t, y) - f(t, y^*)\| \leq \|y - y^*\| \max_B \|f_y(t, y)\| ,$$

for $y, y^* \epsilon B_o$. We must now define what we mean by the norm of a matrix. Let y be an arbitrary vector, such that $\|y\| \leq 1$. Then we define the norm of a matrix M as

$$\|M\| \equiv \max \|My\| . \tag{36}$$

We do this, since we would like

$$\|MN\| \leq \|M\| \cdot \|N\| \Longrightarrow \|My\| \leq \|M\| \cdot \|y\| , \tag{37}$$

where M and N are matrices. Some obvious ways of defining the norm of a matrix do not satisfy this condition, for example $\|M\| = \max |m_{ij}|$ does not. We should show that $\|x\|$, x a vector, has the same value under the definition of norm of a vector and as $n \times 1$ matrix, this is obviously so from Equation (37).

To prove the theorem, we begin with defining

$$y_o(t) = y_o$$

and

$$y_{n+1} = y_o + \oint_{t_o}^{t} f\left(t, y_n(t)\right) \, dt \qquad (n \geq 0, t \in \mathcal{L}) \tag{38}$$

where $\oint$ denotes that, the integration path is along the curve $\mathcal{L}$. Now

$$\|y_{n+1}(t) - y_o\| \leq ML \leq b \quad \text{if} \quad y_n(t) \in B_o(y_o, b) \Rightarrow y_{n+1}(t) \in B_o(y_o, b)$$

Further, all the y_n are continuous on $\mathcal{L}$. We will now prove by induction that

$$\|y_{n+1}(t) - y_n(t)\| \leq M \frac{K^n L(t)^{n+1}}{(n+1)!} \tag{39}$$

a statement which we denote by S_n, and L(t) length of that part of $\mathcal{L}$ from t_o to t. Since

$$\|y_1 - y_o\| \leq \left\| \oint_{t_o}^{t} f(t, y_o) \, dt \right\| \leq ML(t)$$

Equation (39) is true for $n = 0$. We must now show that if S_n, then S_{n+1}. To prove that such is true, consider

$$\|y_{n+1}(t) - y_n(t)\| = \left\| \oint_{t_o}^{t} \left[f\left(t, y_n(t)\right) - f\left(t, y_{n-1}(t)\right) \right] dt \right\| . \tag{40}$$

Now $\mathcal{L}$ is a curve and so by definition it has a parametric representation, $\forall \, t \in \mathcal{L}$, $t = t(\tau)$, $0 \leq \tau \leq 1$. The right hand side of Equation (40) becomes

$$\left\| \int_0^{\tau} [f\left(t(\tau), y(t(\tau))\right) - f\left(t(\tau), y_{n-1}(t(\tau))\right) t'(\tau) \, dt \right\| \equiv \|I\| \quad \text{say} . \tag{41}$$

where the upper limit , τ, of the integration is the value of τ such that $t(\tau) = t$. (This would be less confusing if instead of τ under the integral we used a dummy variable, α , say.

Now we have a real integral and so we can apply the estimate appropriate for the norm of such an integral and obtain

$$\|I\| \leq \int_0^{\tau} \left\| \left[f\left(t, y_n(t)\right) - f\left(t, y_{n-1}(t)\right) \right] t'(\tau) \right\| d\tau$$

where $t = t(\tau)$. Hence applying the Lipschitz condition and S_n,

$$\|I\| \leq K \int_0^{\tau} M \frac{K^{n-1}}{n!} L(\tau)^n \left| t'(\tau) \right| d\tau .$$

But $\frac{dL\left(t(\tau)\right)}{d\tau} d\tau = ds = \left| t'(\tau) \right| d\tau$ so that

$$\|I\| \leq M \frac{K^n}{n!} \int_0^{\tau} L\left(t(\tau)\right) dL\left(t(\tau)\right) = M \frac{K^n}{(n+1)!} L(t)^{n+1} . \tag{42}$$

We have then, that

$$\left\| y_{n+1}(t) - y_n(t) \right\| \leq M \frac{K^n}{(n+1)!} L(t)^{n+1} \tag{43}$$

i.e., S_{n+1}, and our induction proof is complete.

We now use this result to obtain that

$$\sum_{n=0}^{\infty} \left(y_{n+1}(t) - y_n(t) \right) < M \sum_{n=0}^{\infty} \frac{K^n L^{n+1}}{(n+1)!} \tag{44}$$

so that $\sum_{n=0}^{\infty} \left(y_{n+1}(t) - y_n(t) \right)$ is uniformly and absolutely convergent for y_n, $t \in B$. Now consider the partial sums of

$$y(t) \equiv y_o + \sum_{n=0}^{\infty} \left(y_{n+1}(t) - y_n(t) \right) \tag{45}$$

which are just y_k. That Equation (45) is absolutely and uniformly convergent implies that $y_k(t) \longrightarrow y(t)$ uniformly, and so, since $f(t, y)$ is continuous, this implies that $f(t, y_k(t)) \xrightarrow[\text{on } \mathcal{L}]{\text{uniformly}} f(t, y(t))$. Finally, since

$$\lim_{n \to \infty} \int_{\mathcal{L}} g_n(t)\, dt = \int_{\mathcal{L}} \lim_{n \to \infty} g_n(t)\, dt = \int_{\mathcal{L}} g(t)\, dt$$

holds for $g_n(t)$ uniformly convergent to $g(t)$ on $\mathcal{L}$, we have from

$$\lim_{n \to \infty} y_{n+1}(t) = \lim_{n \to \infty} \left(y_o + \int_{\mathcal{L}\, t_o}^{t} f\left(t, y_n(t)\right) dt \right)$$

that

$$y(t) = y_o + \int_{\mathcal{L}\, t_o}^{t} f\left(t, y(t)\right) dt .$$

Since for $t \in \mathcal{L}$ we can write $t = t(\tau)$, $\tau \in R$, the derivative $\dot{y}(t)$ exists and $\dot{y}(t) = f(t, y)$, so our solution satisfies the equation we started with. More generally, if we have

$$\int_{z_o}^{w} f(z)\, dz = g(w), (z, z_o, w \in C),$$

we would require that $f(z)$ be holomorphic on some simply connected domain D for $g'(w) = f(w)$. We do not require this since we confine t to a curve $t = t(\tau)$, $t \in R$.

We now consider the implications of supposing that $f(t, y)$ is holomorphic first of all on $\left\{ \mathcal{L}, \|y - y_o\| \leq b \right\} = B$. B is a closed bounded set and so $f(t, y)$ is bounded and the partials exist and are bounded on B. We can then dispence with the Lipschitz condition and our results hold. Now suppose $f(t, y)$ is holomorphic on $|t - t_o| \leq r$, $\|y - y_o\| \leq b$, then $f(t, y)$ is holomorphic on every curve in $|t - t_o| \leq r$ and for $\|y - y_o\| \leq b$, in particular on every radius of $|t - t_o| \leq r$. Therefore, the solution exists on $|t - t_o| \leq r$ and is holomorphic and therefore is unique. The solution has a power series expansion in $(t - t_o)$, and it is possible to establish this in the form of a Taylor Series which is convergent in the circle $|t - t_o| \leq r$ at least. Further, the solution can be constructed from

$$y_{n+1}(t) = y_o + \int_{t_o}^{t}_{\mathcal{L}} f\big(t, y_n(t)\big)\, dt .$$

We can also use this solution to estimate how far y deviates from y_o in a given time. We have seen that $y_n(t) \subset B_o$, so the solution satisfies $\|y - y_o\| \leq b$ provided $t \in \mathcal{L}$. Now, in general, we want as long a time path as possible, but we have the condition $ML \leq b$ to be satisfied so that we must ensure that M is as small as we can make it, and then we have that b is determined from $ML \leq b$. To ensure that M is small, we subtract off the principal parts as in the perturbation theory of n-body problem. We have then that the solution deviates a distance $\leq ML$ from the initial point; this is the sharpest approximation available.

We shall now consider a set of autonomous differential equations

$$\dot{y} = f(y) .$$

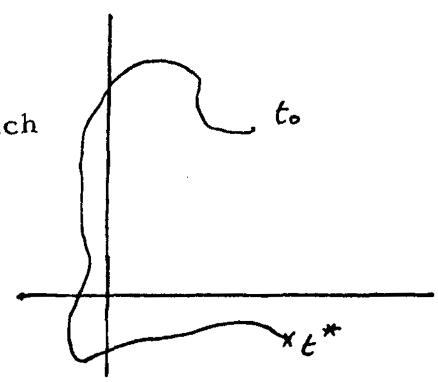
Any non-autonomous system can be written in autonomous form as follows: let $y_{n+1} = t$ and replace y by $\tilde{y} = (y_1, \ldots, y_n, y_{n+1})$ and f by $\tilde{f} = (f_1, \ldots, f_n, 1)$. Then the system of non-autonomous equations becomes

$$\dot{\tilde{y}} = \tilde{f}(\tilde{y}) .$$

Now, consider the system

$$\dot{x} = f(x)$$

with a solution $x = x(t)$ holomorphic on a curve $\mathcal{L}$ less its end point t^*. t_o is the initial point. Let K be the set of all $x = x(t)$, $t \in \mathcal{L} - t^*$. Assume $f(x)$ is holomorphic on a closed and bounded set $s \in C^n$, which contains K, $K \subset s$. Note that K is not a curve under a rigorous definition, since it has no end point. We shall refer to it as a curve for simplicity. K is called the trajectory for the initial



conditions t_o, x_o. The trajectory is contained in the set s. We shall now prove that x(t) is holomorphic at $t = t^*$, hence also holomorphic in a neighborhood of t^* and therefore continuable to t^*. This proposition is known as the Theorem of Poinlevé.

Holomorphic on a set $\Longrightarrow$ holomorphic at each point of the set. To be holomorphic at a point, a function must be continuous, as a function of all variables and the partial derivative of the function with respect to each variable must exist. By continuous as a function of all variables, we mean that $\forall\ \epsilon > 0,\ \exists\ \delta(t, x^*)$ s.t. $\|f(x) - f(x^*)\| < \epsilon$ whenever $\|x - x^*\| < \delta$. It follows that a Taylor series expansion exists in the neighborhood of t^*. which is a multiple power series in the components of x, and each component of f can be expanded in this manner. Since f(x) is holomorphic on S, then to each $\sigma \in S$, we can associate an open ball $B(\sigma, r(\sigma))$: $\|x - \sigma\| < r(\sigma)$, $r(\sigma) > 0$. Further, f(x) is holomorphic in $B(\sigma, r(\sigma))$. We define

$$R \equiv \bigcup_{\sigma \in S} B\left(\sigma,\ r(\sigma)\right),$$

and notice that R is open. We have then an infinite open covering of S and may apply the Heine-Borel theorem to obtain a finite open subcovering. We note that $B\left(\sigma, \frac{1}{2} r(\sigma)\right)$ is also an open covering, and again from this, we may choose a finite subcovering by the Heine-Borel theorem. Let us denote by σ_j the centers of the chosen balls that form this subcovering, $(j = 1, \ldots, N)$, with

$$\bigcup_{j=1}^{N} B\left(\sigma_j, \tfrac{1}{2} r(\sigma_j)\right) \supset S.$$

Now take an arbitrary point $\sigma \in S \Longrightarrow \exists_j,\ 1 \leq j \leq N$ such that

$$\sigma \in B\left(\sigma_j,\ \tfrac{1}{2} r(\sigma_j)\right),$$

which in turn implies

$$B\left[\sigma\ ;\ \tfrac{1}{2} r(\sigma_j)\right] \subset B\left(\sigma_j;\ r(\sigma_j)\right) \subset R\ ,$$

i.e., the closed ball center σ, $\|\sigma - \sigma_j\| < \frac{1}{2} r(\sigma_j)$, of radius $\frac{1}{2} r(\sigma_j)$, is contained in the open ball center σ_j and radius $r(\sigma_j)$. We define

$$r \equiv \min_{j=1,\ldots,N} r(\sigma_j) > 0$$

and take the ball

$$B[\sigma; r] \subset R.$$

f is holomorphic in R and hence is holomorphic in each $B[\sigma; r]$. We have now set up the requirements for our existance theorem to hold. Our r replaces the b of the existance theorem, f(x) is holomorphic in $\|x - \sigma\| \leq r$, $\sigma \in S$, S is closed and bounded. The norm $\|f(x)\| \leq M$ in $\bigcup_{\sigma \subset S} B[\sigma; r]$ so that we may choose $L = r/2M$ to give $ML = r/2 < r(\rightleftarrows b)$.

We now choose $t_1 \in \mathcal{L}$ such that $t_1 \neq t^*$, and $d_{\mathcal{L}}(t_1, t^*) \leq L$. The point $x(t_1) \in S$, and so it exists up to t_1. Using this $x(t_1)$ as $\sigma \subset K \subset S$, the initial conditions or initial point of a solution with t_1 instead of t_o, we may apply our existence theorem and so obtain a solution along $\mathcal{L}$ for a length $L \leq r/2M$. We may choose $L = r/2M$, and so we have continued the solution at least as far as t^* and the theorem stands.

2.3 THE TWO-BODY PROBLEM

In our proof of the existence of periodic solutions to the restricted three-body problem, we shall require what we call "starting solutions." Frequently, such solutions can be obtained by putting the smaller of the primaries equal to zero and thus obtaining the two-body problem. Then by analytical continuation of the solution in μ, the mass of the smaller primary, we obtain periodic solutions for the restricted three-body problem. In the following paragraphs we shall be more explicit about this.

In Euclidean 3 space, the problem of two bodies is defined by the pair of differential equations

$$\begin{aligned} m_1 \ddot{\xi}_1 &= k m_1 m_2 (\xi_2 - \xi_1) r_{12}^{-3} \\ m_2 \ddot{\xi}_2 &= k m_1 m_2 (\xi_1 - \xi_2) r_{12}^{-3} \end{aligned} \tag{46}$$

where, at time t, the point masses m_1, m_2 have position vectors ξ_1, ξ_2, respectively, and $r_{12} = \xi_1 - \xi_2$ is the distance between the two mass points. Adding the pair of Equations (46) leads to the conservation of linear momentum, namely

$$m_1 \ddot{\xi}_1 + m_2 \ddot{\xi}_2 = 0$$

which integrates to

$$m_1 \dot{\xi}_1 + m_2 \dot{\xi}_2 = \text{const.} = \gamma_1 \text{, say .} \tag{47}$$

Integrating again gives

$$m_1 \xi_1 + m_2 \xi_2 = \gamma_1 t + \gamma_2 = m \bar{\xi}$$

where $m = m_1 + m_2$. The motion of the center of mass $\bar{\xi}$ is, then, one of constant velocity on a straight line.

We now transfer coordinate origin to the particle (point mass) m_1. To this end, put

$$\xi = \xi_2 - \xi_1$$

so that we obtain the single equation from Equation (46)

$$\ddot{\xi} = -km \xi r^{-3} \tag{48}$$

where $r = r_{12} = |\xi_1 - \xi_2| = |\xi|$.

The problem defined by Equation (48) is more correctly referred to as the Kepler Problem. The relationship between the solution of the two problems is

$$m \xi_2 = \gamma + m_1 \xi , \qquad m \xi_1 = \gamma - m_2 \xi$$

where $\gamma = \gamma_1 t + \gamma_2$.

Now ξ is the vector (x_1, x_2, x_3), so that $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, and Equation (48) is, in fact, three scalar equations

$$\ddot{x}_j = -\mu x_j r^{-3} \qquad (j = 1, 2, 3)$$

where μ = km, so that the Equation (48) is not trivial. We commence our search for a solution by forming the vector product of Equation (48) with ξ to give

$$\ddot{\xi} \times \xi = 0$$

which integrates to

$$\dot{\xi} \times \xi = \text{const.} = C \tag{49}$$

since $\dot{\xi} \times \dot{\xi} = 0$. Equation (49) is known as Kepler's Law of area. Forming the scalar product of Equation (49) with ξ gives

$$0 = C \cdot \xi \Longrightarrow C \perp \xi \tag{50}$$

on using the cyclic law applicable to the triple scalar product $\dot{\xi} \times \xi \cdot \xi$. Equation (50) states that the motion is perpendicular to a fixed (constant) vector and is, therefore, in a plane.

Now consider

$$\begin{aligned}
\frac{d}{dt}\left(\mu \frac{\xi}{r}\right) &= \mu\left(\dot{\xi} \cdot \text{grad}\, \frac{1}{r}\, \xi + \frac{1}{r}\, \dot{\xi}\right) \\
&= \mu\left(-\frac{1}{r^3} \quad \dot{\xi} \cdot \xi \quad \xi + \frac{1}{r}\, \dot{\xi}\right) \\
&= \mu\left(-\frac{1}{r^3} \quad \dot{\xi} \cdot \xi \quad \xi + \frac{r^2}{r^3}\, \dot{\xi}\right) \\
&= \mu\, \frac{\xi}{r^3} \times \left(\dot{\xi} \times \xi\right) \\
&= C \times \ddot{\xi} = \frac{d}{dt}(C \times \dot{\xi}) .
\end{aligned}$$

Integration gives us immediately

$$\mu \frac{\xi}{r} + \dot{\xi} \times C = \rho = \text{const.} \tag{51}$$

Taking the dot product of this equation with ξ gives

$$\mu r + \xi \cdot \dot{\xi} \times C = \rho \cdot \xi$$

and therefore

$$\mu r - \rho \cdot \xi = C^2 \tag{52}$$

Now, $\xi \cdot \rho = r|\rho| \cos\phi$ and so Equation (52) becomes, on solving for r,

$$r = \frac{C^2}{\mu - |\rho| \cos\phi}$$

$$= \frac{C^2/\mu}{1 - \frac{|\rho|}{\mu}\cos\phi}$$

which is of the form

$$r = \frac{p}{1 - \epsilon \cos\phi} .$$

The equation of the orbit is that of a conic section. If $\epsilon < 1$, r is bounded, giving an ellipse (or in the case $\epsilon = 0$, a circle as a special case); if $\epsilon = 1$, we have a parabola, and for $\epsilon > 1$, a hyperbola. We notice that the coordinate center is at a focus.

Notice that the motion is on a straight line if and only if $\gamma = 0$ since $\dot{\xi} \times \xi = \gamma = 0 \Longrightarrow$ that $\dot{\xi}$ is in same or opposite direction to ξ.

We wish to obtain the position of the particle in its orbit as a function of time. However, this can only be accomplished formally as follows: represent ξ by $r\xi_o$ where $\xi_o^2 = 1$, differentiation of the latter gives $\xi_o \cdot \dot{\xi} = 0$ so that $\dot{\xi} = \dot{r}\xi_o + r\dot{\xi}_o$, and hence

$$\gamma = \dot{\xi} \times \xi = r^2\dot{\xi}_o \times \xi_o$$

(Note, using γ for C of Equation (49),) or

$$|\gamma| = r^2 \left|\dot{\xi}_o\right| . \tag{53}$$

Now we may write

$$\xi_o = u_o \cos\varphi + v_o \sin\varphi = e^{i\varphi}$$

so that $\dot{\xi}_o = i\dot{\varphi}\, e^{i\varphi}$, and $\left|\dot{\xi}_o\right| = \left|\dot{\varphi}\right|$.

Therefore, on substituting into Equation (53), we have

$$|\gamma| = r^2 \left|\dot{\varphi}\right| = \pm\, r^2 \dot{\varphi} .$$

Now assume $\gamma \neq 0$, which implies

$$t = \frac{\pm 1}{|\gamma|} \int_{\psi=\varphi_o}^{\varphi} r^2 \, d\psi$$

and so

$$t = \pm \frac{|\gamma^3|}{\mu^2} \int_{\varphi_o}^{\varphi} (1 - \epsilon \cos\psi)^2 \, d\psi \tag{54}$$

from the interpretation of Equation (52).

The solution is periodic if $0 \leq \epsilon \leq 1$ in which case we can obtain

$$\int_0^{2\pi} (1 - \epsilon \cos\psi)^{-2} \, d\psi = 2\pi(1 - \epsilon^2)^{-3/2}$$

which holds for all complex ϵ such that $\epsilon < 1$. The period T is then

$$T = 2\pi \frac{|\gamma|^3}{\mu^2} (1 - \epsilon)^{-3/2} \tag{55}$$

for elliptic motion.

We define the energy as

$$h \equiv \frac{1}{2}\dot{\xi}^2 - \frac{\mu}{r}$$

where we use $\equiv$ to denote a defining equality. Then

$$\dot{h} = \dot{\xi}\cdot\ddot{\xi} + \frac{\mu}{r^2}\dot{r} = \dot{\xi}\cdot\ddot{\xi} + \frac{\mu}{r^3}\dot{\xi}\cdot\xi$$

since $r = \sqrt{\xi^2}$, and so $\dot{r} = \frac{\xi\cdot\dot{\xi}}{r}$. Now scalar multiplication is distributive over addition so that

$$\dot{h} = \dot{\xi}\cdot\left(\ddot{\xi} + \mu\frac{\xi}{r^3}\right) = 0 \tag{56}$$

which implies that h is a constant along a trajectory. Such a constant of the motion is called an integral, e.g., h is an integral as is each component of $\dot{\xi} \times \xi$. Now on using $\xi = r\xi_o$ and Equation (53), we have that the energy

$$h = \frac{1}{2}\left(\dot{r}^2 + \gamma^2 r^{-2}\right) - \frac{\mu}{r}.$$

Now $h < 0$ implies that the trajectory is bounded for, suppose that it is not, then $h \longrightarrow 1/2\,\dot{r}^2 \geq 0$ for real $\dot{r}$, which is contrary to the supposition that $h < 0$. Hence, $h < 0$ implies that the motion is bounded. The converse is also true, namely, if the motion is bounded, then $h < 0$. To prove this, we consider the two cases: (1) the elliptic case, $\gamma \neq 0$, (2) the linear case, $\gamma = 0$. For case (1), we have $\gamma \neq 0$ and

$$2h = \gamma^2 R^{-2} - \frac{2\mu}{R}$$

also (57)

$$2h = \gamma^2 r^{-2} - \frac{2\mu}{r}$$

where R and r are the maximum and minimum values of r, i.e., when $\dot{r} = 0$. Define $2a = R + r$ and so

$$2\mu\left(\frac{1}{r} - \frac{1}{R}\right) = \gamma^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)$$

or

$$2\mu\left(\frac{R - r}{rR}\right) = \gamma^2 \frac{R^2 - r^2}{r^2 R^2}$$

which implies that

$$2\mu r R = \gamma^2 \, 2a \, . \tag{58}$$

Starting again from Equations (57), we multiply the first by R and the second by r and add to give

$$2L\,(r + R) = \gamma^2 \left(\frac{1}{R} + \frac{1}{r}\right) - 4\mu$$

$$= \gamma^2 \frac{2a}{R} - 4\mu$$

$$= 2\mu$$

on using Equation (58). Now $R + r = 2a$, and so

$$h = -\frac{\mu}{2a} < 0 .$$

Incidentally, we see that the energy is a function of the major axis alone and is independent of the eccentricity, ϵ . For the second case, we have that $\gamma = 0$ and the motion is confined to a straight line, hence

$$h = \frac{1}{2}\dot{r}^2 - \frac{\mu}{r} .$$

Since the motion is bounded, r reaches a maximum value where $\dot{r} = 0$, and hence

$$h = -\frac{\mu}{R} < 0 .$$

From these considerations, we see that the motion is unbounded for $h \geq 0$.

We have already noted that the motion is planar, which allows the motion to be represented in the complex plane. This representation is preferred over two-dimensional vectors for several reasons; one, of course, is the difficulties arising from vector multiplication. However, care must be taken in the complex, for example, $|\xi| \neq \sqrt{\xi^2}$ but $= \sqrt{\xi\bar{\xi}}$. In complex notation, then

$$\xi = x_1 + i x_2$$

and we define a transformation, in the complex, by

$$\xi = u^2 \tag{59}$$

which gives $x_1 = u_1^2 - u_2^2$, $x_2 = 2u_1u_2$ as an equivalent definition of the transformation.

We now define the new time variable s by

$$s = \int_0^t \frac{dt}{|\xi(t)|} \quad . \tag{60}$$

By the superscript $'$ we mean the operator d/ds and by the $\bullet$ over, we mean d/dt. We further define $v = \bar{u}$, so that $r = |\xi| = uv$. Now for any function, $z = z(t)$, $\frac{dz}{ds} = \frac{dz}{dt} t' = \frac{dz}{dt} r$, or $\dot{z} = \frac{1}{r} z'$. Using this, $\dot{\xi} = 2u\dot{u} = 2u\frac{u'}{r} = 2\frac{u'}{v}$, and the second derivative with respect to time

$$\ddot{\xi} = 2\frac{1}{r}\frac{vu'' - u'v'}{v^2}$$

$$= \frac{2vu'' - 2u'v'}{uv^3} \quad .$$

The energy h transforms to

$$h = 2\frac{u'v'}{uv} - \frac{\mu}{uv}$$

or

$$huv = 2u'v' - \mu .$$

The equation of the motion becomes, then,

$$\ddot{\xi} = (2vu'' - \mu - huv)/uv^3$$

$$= -\mu \frac{u^2}{(uv)^3}$$

which implies

$$2uv'' - huv = 0.$$

If we do not have a collision, we can divide by v to give

$$u'' = \frac{h}{2} u \tag{61}$$

which has the solution

$$u = c\cos(\omega s + \beta) \tag{62}$$

where c and β are constants (complex) of integration and $\omega = \sqrt{-h/2}$, which is real if $h \leq 0$, and purely imaginary if $h > 0$. Since c and β are both complex, each is equivalent to two real constants and $\omega = \sqrt{-h/2}$ makes five constants in all. However, only four are needed since Equation (61) is equivalent to two real second-order equations. We must have, then, a functional relationship among h, c and β. This is just the expression we obtained for h, namely,

$$huv = 2u'v - \mu. \tag{63}$$

From Equation (62), we have

$$u' = -c\omega \sin(\omega s + \beta)$$

and

$$v = \bar{u} = \bar{c}\cos(\bar{\omega}s + \bar{\beta})$$

since s is real and ω may be purely imaginary. Also, $v' = \bar{u}' = -\bar{c}\,\bar{\omega}\sin(\bar{\omega}s + \bar{\beta})$. Substituting these expressions into Equation (63) gives

$$\mu = -h\,|c|^2 \begin{cases} \cos(\beta - \bar{\beta}) & \text{if } h < 0 \\ \cos(\beta + \bar{\beta}) & \text{if } h > 0 . \end{cases}$$

This allows the calculation of, say, the real ($h < 0$) or imaginary ($h > 0$) part of β in terms of h and c. We have one case left, $h = 0$.

For the case $h = 0$, we have $u'' = 0$, and so $u = \alpha s + \beta$ and $\xi = (\alpha s + \beta)^2$, $u' = \alpha$, $v' = \overline{\alpha}$, and so $\mu = 2\alpha\overline{\alpha}$ is the required relationship. In the general case, $h \neq 0$, we note that

$$\xi = u^2 = c^2 \cos^2(\omega s + \beta) = \frac{c^2}{2}\Big(1 + \cos(2\omega s + 2\beta)\Big).$$

We notice that the motion is on a straight line when both ω and β are either real or purely imaginary. In the neighborhood of s_1, we may expand ξ as a power series in $(s - s_1)$:

$$\xi = \sum_{n=2}^{\infty} c_n (s - s_1)^n$$

This follows trivially from the fact that cosine has a power series expansion. If $h = 0$, the series reduces to a single term of the 2^{nd} degree.

We have, now, a solution as a function of s that holds for all $u \neq 0$, or equivalently all $\xi \neq 0$. Our problem is this, can we analytically continue the solution to $\xi = 0$. We can mathematically, but physically, we have lost the meaning of the solution.

We proceed from Equation (62), remembering that $\xi = u^2$. For $u = 0$, and hence $\xi = 0$, we must have

$$\omega s_1 + \beta = \frac{\pi}{2}$$

and the expansion of u becomes

$$u = \sum_{n=1}^{\infty} u_n (s - s_1)^n = (s - s_1) \sum_{n=0}^{\infty} \tilde{u}_{2n} (s - s_1)^{2n}.$$

This is so since $\cos(\pi/2 - s) = \sin s$. On writing

$$\xi = (s - s_1)^2 \sum_{n=0}^{\infty} c_n (s - s_1)^{2n}$$

we see that ξ has a zero of second order at $s = s_1$.

Now we defined $s = \int_0^t \frac{dt}{r}$ so that $t = \int_0^s r\,ds = \int_0^s u\,v\,ds$. We must determine the behavior of the integral for t near s_1. Now, $t_1 = \int_0^{s_1} u\,v\,ds$, and so

$$t - t_1 = \int_{s_1}^{s} u\,v\,ds = \int_{s_1}^{s} (s - s_1)^2 \sum_{n=0}^{\infty} a_n (s - s_1)^{2n}\,ds$$

which, since it is a power series, it is uniformly convergent within its radius of convergence and can be integrated termwise to give

$$t - t_1 = (s - s_1)^3 \sum_{n=0}^{\infty} b_n (s - s_1)^{2n}.$$

This can be inverted, formally, to give

$$s - s_1 = \sum_{n=1}^{\infty} d_n (t - t_1)^{n/3}.$$

This is a quasi-power series in $t - t_1$, which becomes, on putting $(t - t_1)^{1/3} = z$, a power series in z. Hence

$$\xi = \sum_{n=2} e_n (t - t_1)^{n/3}.$$

We see then that ξ is continuous through the collision at time $t = t_1$, since for a collision $\xi = 0$. Although we have found that the solution to the Kepler problem is continuably through a collision, this function cannot be said to satisfy the Kepler equation at $\xi = 0$. In fact we do not claim this, but just that the solution as a function in its own right is continuable through $\xi = 0$.

We notice that $(t - t_1)^{1/3}$ is continuous at t_1 if considered as a function of a real variable, however, when considered in the complex, it has a singularity at $t = t_1$. From the known behavior of $\sqrt[3]{w}$ at the origin, we recognize the singularity as a branch point of order 2, in fact, then, an algrbraic branch point. Notice that the derivatives, $z = 1/3\ (t - t_1)^{-2/3}$, of $z = (t - t_1)^{1/3}$ with respect to time t becomes unbounded at $t = t_1$. The time derivative of ξ is

$$\xi = \frac{2}{3}\, e(t - t_1)^{-1/3}\, t + \ldots$$

which is singular at $t = t_1$.

2.4 THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM

The equations of motion of the three-body problem are given immediately by Equation (4) on putting $n = 3$; at the same time, we change notation to $g_k = x_k$ $(k = 1, 2, 3)$ to give

$$\ddot{q}_k = \sum_{j \neq k} \frac{m_1}{r_{jk}^3} (q_j - q_k) \qquad (k = 1, 2, 3).$$

Now the equation for $k = 3$ is

$$\ddot{q}_3 = \frac{m_1}{r_{13}^3}(q_1 - q_3) + \frac{m_2}{r_{23}^3}(q_2 - q_3) \tag{64}$$

which we notice is invariant if we allow $m_3 \to 0$. In the limit, the equations for $k = 1, 2$, become

$$\ddot{q}_1 = \frac{m_2}{r_{12}^3}(q_2 - q_1)$$

$$\ddot{q}_2 = \frac{m_1}{r_{12}^3}(q_1 - q_2) \tag{65}$$

respectively, which are the two-body equations of motion. We may choose as the solution of Equations (65) any of the Keplerian orbits but restrict ourselves to elliptic and circular since these are non-degenerate periodic solutions. If an elliptic orbit is chosen for the relative motion between m_1 and m_2, then we have the elliptic restricted three-body problem, and if the relative motion is circular, then we name this the restricted three-body problem. We shall develop the elliptic case and, when appropriate, reduce it to the circular case.

Our first step is to normalize the mass so that $m_1 + m_2 = 1$ and so obtain

$$\ddot{q}_o = -q_o \left|q_o\right|^{-3} \tag{66}$$

where $q_0 = q_2 - q_1$, $q_1 = -m_2 q_0$, $q_2 = m_1 q_0$, $m_1 + m_2 = 1$, and the origin of coordinates is at the center of mass. For the third body, the one of insignificant mass, we have

$$\ddot{q} = \frac{m_1}{r_1^3}(q_1 - q) + \frac{m_2}{r_2^3}(q_2 - q) \tag{67}$$

We represent q_0 by

$$q_0 = \frac{1 - \epsilon^2}{1 - \epsilon \cos s}\, e^{is} \tag{68}$$

where

$$t = (1 - \epsilon^2)^{3/2} \int_o^s (1 - \epsilon \cos \sigma)^{-2}\, d\sigma,$$

which we now show is a solution of the two-body problem. We define

$$z_1 = \mathscr{I}m\, \bar{q}_o \dot{q}_o, \quad z_2 = |Z|, \quad z_3 = \operatorname{arc} q_o, \quad z_4 = -\operatorname{arc} Z \tag{69}$$

where

$$Z = q_o \left|q_o\right|^{-1} + i z_1 \dot{q}_o .$$

Further, by r_o we denote $\left| q_o \right|$ and so

$$\bar{q}_o Z = r_o + i z_1 \bar{q}_o \dot{q}_o$$

and so $\mathrm{Re}(\bar{q}_o Z) = r_o - z_1^2$. Now, from the definitions of z_2 and z_4, we have

$$Z = z_2 e^{-i z_4}$$

and so

$$z_1^2 = r_o \left(1 - z_2 \mathrm{Re}\, e^{-i z_3 - i z_4}\right)$$

$$= r_o \left(1 - z_2 \cos(z_3 + z_4)\right) .$$

Solving for r_o gives

$$r_o = z_1^2 \left(1 - z_2 \cos(z_3 + z_4)\right)^{-1} . \tag{70}$$

Using this expression for r_o and solving the definition of Z for $\dot{q}_o$ with the appropriate substitutions gives

$$\dot{z}_1 = \mathscr{I}m\,(\dot{\bar{q}}_o \dot{q}_o + \bar{q}_o \ddot{q}_o) = \mathscr{I}m\, \bar{q}_o (-q_o r_o^{-3}) = 0 \tag{71}$$

i.e., z_1 is constant.

Now,

$$\bar{q}_o \dot{q}_o = q_o (\dot{r}_o + i r_o \dot{z}_3) e^{i z_3}$$

$$= r_o \dot{r}_o + i r_o^2 \dot{z}_3 .$$

Taking the imaginary part gives

$$z_1 = r_o^2 \dot{z}_3$$

$$\therefore \dot{z}_3 = z_1 r_o^{-2} .$$

Now

$$\dot{Z} = i\dot{z}_3 e^{iz_3} + iz_1 r_o^{-2}$$

$$= iz_1 r_o^{-2} e^{iz_3} + iz_1 (-e^{iz_3} r_o^{-2}) = 0 \; .$$

But $Z = z_2 e^{-iz_4}$, hence,

$$\dot{z}_2 = 0, \quad \dot{z}_4 = 0 \quad . \tag{72}$$

We now introduce

$$s = \int_o^t z_1 r_o^{-2} \, d\tau \tag{73}$$

and so $z_3' = \dot{z}_3 t' = z_1 r_o^{-2} = z_1 r_o^{-2} r_o^2 z_1^{-1} = 1$.

We now denote the initial value of z by $\mathring{z}$, so we may write, from Equations (66), (67) and (68)

$$z_1 = \mathring{z}_1, \quad z_2 = \mathring{z}_2, \quad z_3 = s + \mathring{z}_3, \quad z_4 = \mathring{z}_4 .$$

We take $\mathring{z}_3 = \mathring{z}_4 = 0$ and $z_2 = \epsilon$, $0 < \epsilon < 1$. Using these values to calculate $r_{o_{min}}$ and $r_{o_{max}}$ gives

$$r_{o_{max}} = \mathring{z}_1^2 (1 - \epsilon)^{-1}, \quad r_{o_{min}} = \mathring{z}_1^2 (1 + \epsilon)^{-1} \quad .$$

But $r_{o_{max}} + r_{o_{min}} = 2a_o$ and so

$$a_o = \mathring{z}_1^2 (1 - \epsilon^2)^{-1} .$$

The representation we have chosen for our solution requires that $a_o = 1$, so we must choose

$$\mathring{z}_1^2 = (1 - \epsilon^2).$$

We see then that

$$q_o = r_o e^{iz_3} = \frac{(1 - \epsilon^2)}{1 - \epsilon \cos s} e^{is} .$$

To complete the proof that our chosen representation is a solution, we need an expression for t. We have, on differentiating Equation (70),

$$t' = \frac{1}{\dot{s}} = r_o^2 z_1^{-1}$$

and so

$$t = \overset{\circ}{z}{}_1^3 \int_o^s (1 - \epsilon \cos\sigma)\, d\sigma .$$

i.e.,

$$t = (1 - \epsilon^2)^{1/2} \int_o^s (1 - \epsilon \cos\sigma)^{-2}\, d\sigma , \tag{75}$$

as we required.

The next task is to find a coordinate system in which both m_1 and m_2 are at rest. We choose x_1, x_2 as coordinates in the plane of motion of the massive bodies, and put

$$q_j = x_1^{(j)} + i x_2^{(j)} \qquad (j = 1, 2) \quad .$$

Then

$$r_j = |q - q_j| = \sqrt{\left\{|x - q_j|^2 + x_3^2\right\}} \tag{76}$$

where x_3 forms a mutually perpendicular right triad with x_1 and x_2. What we have done then is put $x = x_1 + i x_2$ for the first two coordinates of $q = (x_1, x_2, x_3)$. The now complex numbers q_1, q_2 have been computed from q_o of Equation (67).

We now write the three scalar equations from 3 in Lagrangian form. To this end, we define

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) + \frac{m_1}{r_1} + \frac{m_2}{r_2} \tag{77}$$

which gives

$$\frac{d}{dt} L_{\dot{x}_k} = L_{x_k} \qquad (k = 1, 2, 3) \tag{78}$$

as the equations of motion, where $r_1 = \left\{\left|x + m_2\right|^2 + x_3^2\right\}^{1/2}$ and $r_2 = \left\{\left|x - m_1\right|^2 + x_3^2\right\}^{1/2}$. We verify this by direct substitution from Equation (77).

We have that $L_{\dot{x}_k} = \dot{x}_k$ and $\therefore$ LHS of Equation (77) is just $\ddot{x}_k$.

$$\therefore \ddot{x}_k = L_{x_k} = -\frac{m_1}{r_1^3}\left(x_k - x_k^{(1)}\right) - \frac{m_2}{r_2^3}\left(x_k - x_k^{(2)}\right) \quad (k = 1,2)$$

which is just the k^{th} $(k = 1,2)$ component of Equation (64), and

$$\ddot{x}_3 = L_{x_3} = -\frac{m_1}{r_1^3} x_3 - \frac{m_2}{r_2^3} x_3 .$$

We must now apply the transformation theory that we have studied. First we note the explicit dependence of L on t through the q_o. We introduce the transformation

$$x = q_o \xi, \qquad x_3 = r_o \xi_3$$

where $\xi = \xi_1 + i\xi_2$. We define L^* by

$$L^* = L^*(\xi, \dot{\xi}, t) = L\left(x(\xi), x(\xi, \dot{\xi}), t\right) = L\left(q_o \xi, (q_o \xi), t\right).$$

Now from Equation (77)

$$L = \frac{1}{2}\left\{\left|\dot{q}_o \xi + q_o \dot{\xi}\right|^2 + \left(\dot{r}_o \xi_3 + r_o \dot{\xi}_3\right)^2\right\} + \frac{1}{r_o} U$$

where, recall, $r_o = |q_o|$ and, now,

$$U = \frac{m_1}{\sqrt{(\xi_1 + m_2)^2 + \xi_2^2 + \xi_3^2}} + \frac{m_2}{\sqrt{(\xi_1 - m_1)^2 + \xi_2^2 + \xi_3^2}} .$$

The purpose of this transformation was to bring the two massive particles, m_1, m_2, to rest. Now from Equations (66), we have

$$q_1 = -m_2 q_o, \quad q_2 = m_1 q_o$$

and so $q_o \xi_1 = -m_2 q_o$, $q_o \xi_2 = m_1 q_o$ yielding $\xi_1 = -m_2$, $\xi_2 = m_1$. We have then fixed the massive particles and so we have fixed the singularities. This transformation may be looked upon as transforming the relative motion of m_1 and m_2 to an equilibrium motion. Geometrically, we now have a pulsating and a non-uniformly rotating coordinate system.

Finally, we perform a time transformation to s by using

$$t' = r_o^2 z_1^{-1} = \rho > 0$$

which also defines ρ. Notice that

$$\frac{\rho}{z_1} = \frac{\rho^2}{r_o^2} = \frac{r_o^2}{z_1^2}$$

and that we denote dependence on s by $\sim$ and by $'$ we mean d/ds, so that

$$\zeta(t) = \tilde{\zeta}(s); \qquad \dot{\zeta} = \tilde{\zeta}' \frac{1}{\rho}$$

where ζ has components ξ_1, ξ_2, ξ_3.

The new Lagrangian is

$$\tilde{L} = \frac{\rho}{z_1} L^* \left(\tilde{\zeta}, \frac{1}{\rho} \tilde{\zeta}', t(s)\right) = \tilde{L} (\tilde{\zeta}, \tilde{\zeta}', s) ,$$

and on performing the necessary operations, we obtain

$$\frac{d}{ds}\tilde{L}_{\xi_k'} = \frac{d}{ds}\left\{\frac{1}{z_1} L^*_{\dot{\xi}_k}\right\} = \frac{\zeta}{z_1}\frac{d}{dt} L^*_{\dot{\xi}_k}$$

$$= \frac{\rho}{z_1} L^*_{\xi_k} = \tilde{L}_{\tilde{\xi}_k} .$$

Our particular choice of Lagrangian has preserved the form of the Lagrange equations. Explicitly our new Lagrangian is

$$\tilde{L} = \frac{\rho}{2z_1}\left\{\left|q_o'\xi + q_o\xi'\right|^2 \frac{1}{\rho^2} + (r_o'\xi_3 + r_o\xi_3')^2 \frac{1}{\rho^2}\right\} + \frac{\rho}{z_1 r_o} U . \tag{79}$$

We now put

$$\frac{\rho}{2r_o} = \alpha = (1 - \epsilon\cos\varphi)^{-1}$$

a function of s. We bring αU to the left hand side of Equation (79), and then expand the right hand side, remembering that q_o and ξ are complex numbers, to give

$$\tilde{L} - \alpha U = \frac{1}{2r_o^2}\left\{\left|q_o'\right|^2\left|\xi\right|^2 + \left|q_o\right|^2\left|\xi'\right|^2 + 2\mathrm{Re}\,\bar{q}_o q_o'\bar{\xi}'\xi + r_o'^2\xi_3^2\right.$$

$$\left. + r_o^2\xi_3'^2 + 2r_o r_o'\xi_3\xi_3'\right\}$$

$$= \left\{\frac{1}{2}\left|\xi'\right|^2 + \xi_3'^2\right\} + \frac{1}{2}\left|\frac{q_o'}{r_o}\right|^2\left|\xi\right|^2 + \frac{1}{2}\left(\frac{r_o'}{r_o}\right)^2\xi_3^2$$

$$+ \frac{r_o'}{r_o}\left(\xi_3\xi_3' + \xi_1\xi_1' + \xi_2\xi_2'\right)$$

$$+ \xi_1\xi_2' - \xi_2\xi_1' .$$

We have calculated $\mathrm{Re}\,\bar{q}_o q_o'\bar{\xi}'\xi$ in the following way. Recall that

$$\bar{q}_o \dot{q}_o = r_o \dot{r}_o + i\, r_o^2 \dot{z}_3$$

since $q_o = r_o e^{iz_3}$, and $z_3 = s$. Introducing ρ into this equation gives

$$\bar{q}_o \dot{q}_o = r_o \dot{r}_o + i\, r_o^2 \rho^{-1}$$

which, on multiplying by ρ gives, on the left hand side, $\bar{q}_o q_o'$, and for the first term on the right hand side, $r_o r_o'$. Using this gives

$$\mathrm{Re}\, \bar{q}_o q_o' \xi \bar{\xi}' = r_o r_o' \,\mathrm{Re}\, \bar{\xi} \xi' + r_o^2 \,\mathcal{I}m\, \bar{\xi} \xi' .$$

Recall that $\xi = \xi_1 + i \xi_2$ and hence $\xi' = \xi_1' + i\, \xi_2'$ and $\bar{\xi} = \xi_1 - i\, \xi_2$, which allows us to write

$$\mathrm{Re}\, \bar{q}_o q_o' \xi \bar{\xi}' = r_o r_o' (\xi_1 \xi_1' + \xi_2 \xi_2') + r_o^2 (\xi_1 \xi_2' - \xi_2 \xi_1') ,$$

which we have used in Equation (80). Expanding the expressions on the right hand side of Equation (80) even further, gives

$$\begin{aligned} \tilde{L} - \alpha U = {} & \frac{1}{2} (\xi_1'^2 + \xi_2'^2 + \xi_3'^2) + \xi_1 \xi_2' - \xi_2 \xi_1' + \frac{1}{2} |\xi|^2 \\ & + \frac{1}{2} \left(\frac{r_o'}{r_o}\right)^2 (\xi_1^2 + \xi_2^2 + \xi_3^2) \\ & + \frac{r_o'}{r_o} (\xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3') \quad . \end{aligned} \tag{81}$$

Now, from $\ddot{q}_o = -q_o r_o^{-3}$, we obtain the first integral

$$\frac{1}{2} |\dot{q}_o|^2 - \frac{1}{r_o} = h_o \quad .$$

Now, $q_o' = (r_o' + i\, r_o)\, e^{is}$, so that

$$h_o = \frac{1}{2\rho}(r_o'^2 + r_o^2) - \frac{1}{r_o}$$

and finally on multiplying by ρ^2/r_o^2, we have, on rearranging and transporting

$$\frac{1}{2}\frac{r_o'^2}{r_o^2} = \frac{\rho^2}{r_o^2} h_o - \frac{1}{2} + \frac{\rho^2}{r_o^3}. \tag{82}$$

But we have established in Section 2.3, that $h = -\frac{1}{2a_o} = \frac{1}{2}$ in our case. We now put

$$R = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

and so (83)

$$R' = \xi_1\xi_1' + \xi_2\xi_2' + \xi_3\xi_3'.$$

Using Equations (82) and (83) allows us to write Equation (81) as

$$L - \alpha(U + \tfrac{1}{2}R) = \frac{1}{2}\left\{\left(\frac{r_o}{z_1^2} - \frac{r_o^2}{z_1^2}\right)R + \frac{1}{2}\frac{r_o'}{r_o}R'\right\} + \frac{1}{2}v^2 + \xi_1\xi_2' + \xi_2\xi_1' - \frac{1}{2}\xi_3^2.$$

We would like to drop the term $\frac{1}{2}\left\{\left(\frac{r_o}{z_1^2} - \frac{r_o^2}{z_1^2}\right)R + \frac{1}{2}\frac{r_o'}{r_o}R'\right\}$ from the Lagrangian. To do so, we need the following theorem:

Lagrangians differing only by the total time derivative of a well behaved function of x and t give rise to equivalent Lagrangian systems. This may be looked upon as a uniqueness theorem. We must now prove this.

Consider two Lagrangians, $f(x, \dot{x}, t)$ and $g(x, \dot{x}, t)$, differing by a total time derivative $\dot{h}(x, t)$. We may write $f = g + \dot{h}$. Now

$$\dot{h} = \sum_{j=1}^{n}(h_{x_j}\dot{x}_j) + h_t,$$

and so

$$L_{x_k} f = L_{x_k} g + \frac{d}{dt}\left(h_{\dot{x}_k}\right) - \sum_{j=1}^{n} h_{x_j x_k} \dot{x}_j - h_{t x_k}$$

$$= L_{x_k} g + \frac{d}{dt}\left(h_{x_k}\right) - \sum_{j=1}^{n} h_{x_j x_k} \dot{x}_j - h_{t x_k}$$

$$= L_{x_k} g + \sum_{j=1}^{n} h_{x_k x_j} - \sum_{j=1}^{n} h_{x_j x_k} \dot{x}_j + h_{x_k t} - h_{t x_k}$$

$$= L_{x_k} g$$

provided the order of differentiation can be reversed.

To drop the expression $\frac{1}{2}\left\{\left(\frac{r_o}{z_1^2} - \frac{r_o^2}{z_1^2}\right) R - \frac{1}{2}\frac{r_o'}{r_o} R'\right\}$, we must show that it is the total time derivative of some function of ζ and s for these correspond to the x and t of our theorem. Now, factoring out the r_o/z_1^2 from the first term gives

$$\frac{r_o}{2z_1^2}(1 - r_o) R + \frac{r_o'}{2r_o} R' = \frac{r_o}{2z_1^2}(1 - r_o)(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

$$+ \frac{r_o'}{r_o}(\xi_1 \xi_1' + \xi_2 \xi_2' + \xi_3 \xi_3')$$

on substituting for R and R'. If the above expression is to be h' then we must have

$$h_{\xi_k} = \frac{r_o'}{r_o} \xi_k , \qquad h_s = \frac{(1 - r_o) r_o}{2z_1}(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

and hence

$$h \equiv \frac{r_o'}{2r_o}(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

For this to be so, we need

$$\left(\frac{r_o'}{r_o}\right)' = \frac{r_o}{z_1^2}(1 - r_o).$$

To show that this is so, we note that the energy intergal, Equation (82), may be written as

$$\frac{1}{2}\dot{r}_o^2 + \frac{1}{2}\frac{z_1^2}{r_o^2} - \frac{1}{r_o} = h_o = -\frac{1}{2}.$$

Recall that

$$t' = \rho = \frac{r_o^2}{z_1}$$

so that

$$\dot{r}_o^2 = \left(r_o' \frac{1}{t'}\right)^2 = \left(\frac{r_o' z_1}{r_o^2}\right)^2.$$

On using this and multiplying by r_o^2/z_1^2, the energy integral becomes

$$\frac{1}{2}\left(\frac{r_o'}{r_o}\right)^2 + \frac{1}{2} - \frac{r_o}{z_1^2} = -\frac{1}{2}\frac{r_o^2}{z_1^2}.$$

We have then as our final Lagrangian

$$\tilde{\tilde{L}} = \frac{1}{2}v^2 + \xi_1\xi_2' - \xi_2\xi_1' - \frac{1}{2}\xi_3^2 + \alpha(U + \frac{1}{2}R)$$

where $v^2 = \xi_1'^2 + \xi_2'^2 + \xi_3'^2$.

We now transform to the Hamiltonian form in precisely the way described in Section 2.1. Our interest lies in periodic solutions in the plane of motion of the massive bodies. We make the restriction to two dimensions on putting $\xi_3 = \xi_3' = 0$. As before, the canonical momentum is derived to be

$$\eta_1 = \tilde{\tilde{L}}_{\xi_1'}(\xi, \xi', s) = \xi_1' - \xi_2$$

$$\eta_2 = \tilde{\tilde{L}}_{\xi_2'}(\xi, \xi', s) = \xi_2' + \xi_1 .$$

We solve this pair of equations for ξ_1', ξ_2', so that

$$\xi_1' = \eta_1 + \xi_2, \qquad \xi_2' = \eta_2 - \xi_1 .$$

Following the scheme given in Section 2.1, we put

$$H = \sum_{k=1}^{2} \xi_k' \eta_k - \tilde{\tilde{L}}$$

and so

$$H = \frac{1}{2}\left[(\eta_1 + \xi_2)^2 + (\eta_2 + \xi_1)^2 - \alpha\left(u + \frac{1}{2}(\xi_1^2 + \xi_2^2)\right)\right] .$$

Our equations of motion are now obtainable as

$$\xi_k' = H_{\eta_k}, \qquad \eta_k' = -H_{\xi_k} \qquad (k = 1, 2) .$$

This is as far as we need take the elliptic restricted three-body problem, for at this point, we are ready to discuss the existence of periodic solutions.

2.5 PERIODIC SOLUTIONS OF THE RESTRICTED THREE-BODY PROBLEM

To reduce the elliptic restricted three-body problem to the (circular) restricted problem, we set $\epsilon = 0$, which yields $\alpha(s) \equiv 1$ and $t = s$. The Hamiltonian is

$$H = \frac{1}{2}\left[(y_1 + x_2)^2 + (y_2 - x_1)^2\right] - U - \frac{1}{2}(x_1^2 + x_2^2) \qquad (84)$$

where $U = \dfrac{m_1}{|x + m_2|} + \dfrac{m_2}{|x - m_1|}$. The equations of motion are of the Hamiltonian form, namely,

$$\dot{x}_k = H_{y_k}, \quad \dot{y}_k = -H_{x_k} \qquad (k = 1, 2). \tag{85}$$

Now, if we put $x = x_1 + i x_2$ $(i^2 = -1)$, we may write Equation (85) as

$$\ddot{x} + 2i\dot{x} + x = -m_1 \frac{x + m_2}{|x + m_2|^3} - m_2 \frac{x - m_1}{|x - m_1|^3}. \tag{86}$$

It is periodic solutions of this equation that are of interest. We must state what we mean by a periodic solution. Formally, we call a function $f(t)$ periodic in t if for some τ, $f(t + \tau) = f(t)$ for all t. Later, we shall see that there are equivalent conditions for periodicity that are more useful for our purpose. We shall prove and use the following periodicity conditions, that if $f(o) = f(\tau)$ and $f'(o) = f'(\tau)$ then $f(t + \tau) = f(t)$ for all t. The converse also holds, so our periodicity condition is equivalent to our definition. Further, to bring our notation in line with that of current literature, we put $m_2 = \mu$, $m_1 = 1 - \mu = \mu^*$.

We now survey briefly the classes of periodic solutions known to exist. These solutions are not necessarily known even in implicit form, but just that such solutions do in fact exist.

1. The Libration Points. These are the equilibrium solutions of Equation (84) and are readily shown to be five in number; three on the real axis separated by m_1 and m_2, and one each side of the real axis, forming an equilateral triangle with m_1 and m_2.

2. Motions near the triangular libration points. Eigenvalues of the matrix of second partials evaluated at the appropriate libration point are all purely imaginary.

3. Motions near the libration points on the real axis. Here, at least one eigenvalue is real, and at least one is imaginary. These libration points are, then, unstable, but some periodic solutions exists for properly chosen initial conditions.

4. $|x| \gg 1$. Intuitively, we suspect that periodic solutions do exist, since we may look upon this case as a perturbation of the two-body problem. The existence of almost circular orbits can be proved.

5. P_3 very near m_1 or m_2, again, can be regarded as a perturbation of the two-body problem.

6. $m_2 \ll 1$. This is almost the two-body problem, even for P_3 near m_2. We call this case the planetary case of the restricted three-body problem, with $m_1 \equiv$ Sun, $m_2 \equiv$ Jupiter, and P_3 being some other planet. If P_3 remains close to m_2, we call this the lunar case. Not only do periodic solutions exist, but also almost periodic solutions, but the proof is possibly somewhat more difficult.

We shall prove the existence of periodic solutions for the cases of most interest to us. These are classes 5) and 6). We shall now prove the existence of periodic orbits of class 6).

Out first step is to move the origin to one of the masses. We choose m_1 and define

$$u = x - m_2,$$

and put $m_2 = \mu$, $m_1 = 1 - \mu = \mu^*$. Substituting into Equation (86) gives

$$\ddot{u} + 2i\dot{u} - u + \mu^* F(u) = \mu\left(1 - F(1 + u)\right) \tag{87}$$

where $F(u) \equiv u/|u|^3$. We call the right hand side of Equation (87) the disturbing "force" or function. If $\mu = 0$, then, the right hand side is zero. We notice also that the left hand side $\longrightarrow \infty$ as $u \longrightarrow 0$.

As we have indicated earlier, a more suitable periodicity condition must be introduced. The periodicity condition that we require is that if a solution cuts the ξ axis orthogonally at two distinct times, then, the solution is periodic. We must now prove this. We choose the axes and our time origin, so that P_3 lies on the ξ axis at a perpendicular crossing. At $t = 0$, we have that

$$z(t) = \bar{z}(t), \quad \dot{z}(t) = -\dot{\bar{z}}(t). \tag{88}$$

Now, if at some later time, $t_o > 0$, Equation (88) holds, we shall prove that the solution is periodic. We introduce

$$V(t) = \bar{z}(2t_o - t), \tag{89}$$

and so,
$$\dot{V}(t) = -\dot{\bar{z}}(2t_o - t).$$

We now wish to show that our differential Equation (86) becomes

$$\ddot{V}(t) + 2i\dot{V}(t) + \mu^* F(V) = \mu\Big(1 - F(1 + V)\Big). \tag{90}$$

Using Equation (89) and conjugating this reduces to

$$\ddot{\bar{z}}(2t_o - t) - 2i\dot{\bar{z}}(2t_o - t) - \bar{z}(2t_o - t) + \mu^* F\Big(\bar{z}(2t_o - t)\Big)$$
$$= \mu\Big(1 - F\Big(1 + \bar{z}(2t_o - t)\Big)\Big).$$

Now
$$V(t_o) = \bar{z}(t_o) = z(t_o), \quad \dot{V}(t_o) = -\dot{\bar{z}}(t_o) = \dot{z}(t_o)$$

where we must consider $V(t)$, $z(t)$ solutions of the appropriate differential equations, and coinside in position and velocity at one time point. Hence, $V(t) = z(t)$ identically, and

$$z(2t_o) = \bar{z}(o) = z(o). \tag{91}$$

By a similar argument, we obtain that

$$\dot{z}(t) = -\dot{\bar{z}}(2t_o - t)$$

and, therefore,

$$\dot{z}(2t_o) = -\dot{\bar{z}}(o) = \dot{z}(o). \tag{92}$$

Since the differential equation is autonomous, $z(2t_o + t) = z(t)$, and so the period is $2t_o$. If we have two perpendicular crossings of the real axis, then the solution is periodic, and we have

$$z(-t) = \bar{z}(t),$$

and we see that the trajectory is symmetric about the real axis. We have only a sufficient condition.

Now, z is a function of μ, and, of course, of the initial conditions. We may show this function dependence, and at $t = t_o$, we have

$$\mathcal{I}m\, z(t_o, z_o, \dot{z}_o, \mu) = 0 = \mathrm{Re}\, \dot{z}(t_o, z_o, \dot{z}_o, \mu).$$

Out intent is to continue analytically the Keplerian solutions obtained for $\mu = 0$, so first we ask: Does the Kepler problem admit periodic solutions with two perpendicular crossing? It does, and in particular, for suitably chosen axes, elliptic orbits have two perpendicular crossings. This elliptic case for $\mu = 0$ is the one we require.

The coordinate system that we have is not practical, so first we put

$$z = \mu^{*\,1/3} x$$

and obtain

$$\ddot{x} + 2i\dot{x} - x + F(x) = \mu\mu^{*\,1/3}\left(1 - F(1 + \mu^{*\,1/3} x)\right) \equiv Q(x) \qquad (93)$$

If we now put $\mu = 0$, the left hand side is just the Kepler problem. As we shall now see, the Kepler motion is simpler.

We define z_1 through z_4 as follows:

$$z_1 = \mathcal{I}m\, \bar{x}(\dot{x} + ix), \quad z_2 = |Z|, \quad z_3 = \arc x + z_4, \quad z_4 = -\arc Z \qquad (94)$$

where $Z \equiv x r^{-1} + i z_1 (\dot{x} + i x)$ and $r = |x|$. These variables are called the elliptic elements. We now wish to express x and $r = |x|$ in terms of z_1 through z_4. Multiplying Z by x gives

$$\bar{x} Z = r + i z_1 \bar{x} (\dot{x} + i x).$$

From the definitions

$$x = r e^{i(z_3 - z_4)} \tag{95}$$

so we may write

$$\operatorname{Re} \bar{x} Z = r - z_1^2$$

and so

$$r = \frac{z_1^2}{1 - \operatorname{Re}\left(e^{i(z_4 - z_3)}\right)} = z_1^2 (1 - z_2 \cos z_3). \tag{96}$$

Now

$$\dot{z}_1 = \mathcal{I}m\, \bar{x}(\dot{x} + i x) + \mathcal{I}m\, \bar{x}(\ddot{x} + i \dot{x})$$

$$= \mathcal{I}m\, (i \dot{\bar{x}} x + \bar{x} \ddot{x} + i \bar{x} \dot{x}) = \mathcal{I}m\, \bar{x} (\ddot{x} + 2 i \dot{x})$$

since $\quad -\mathcal{I}m\, (-i \dot{x} \bar{x}) = \mathcal{I}m\, (i \dot{\bar{x}} x) = \mathcal{I}m\, i \dot{x} \bar{x}$

Now, since $\bar{x} x$ and $\bar{x} F(x)$ are both real, we may write

$$\dot{z}_1 = \mathcal{I}m\, \bar{x}(\ddot{x} + 2 i \dot{x} - x + F(x)) = \mathcal{I}m\, \bar{x} Q$$

from Equation (93). Differentiating Equation (94) gives

$$\dot{x} = (\dot{r} + i r (\dot{z}_3 - \dot{z}_4))\, e^{i(z_3 - z_4)}$$

and so

$$\bar{x} \dot{x} = r \dot{r} + i r^2 (\dot{z}_3 - \dot{z}_4),$$

and

$$\mathcal{I}m\, x\dot{x} = r^2(\dot{z}_3 - \dot{z}_4) = z_1 - r^2$$

from the definition of z_1 and r. Multiplying throughout by r^{-2} gives

$$\dot{z}_3 - \dot{z}_4 = z_1 r^{-2} - 1.$$

Now, on using $x = r\,e^{i(z_3 - z_4)}$, we obtain eventually that

$$\dot{Z} + iZ = iz_1 Q + i(\dot{x} + ix)\dot{z}_1$$

$$= (\dot{z}_2 + iz_2 - iz_2\dot{z}_4)\,e^{-iz_4} \tag{97}$$

since $Z = z_2 e^{-iz_4}$. We now define

$$S \equiv iz_1 Q + i(\dot{x} + ix)\,\dot{z}_1.$$

Multiplying Equation (97) by e^{iz_4} and taking real and imaginary parts gives

$$\dot{z}_2 = \mathrm{Re}\left\{S\,e^{iz_4}\right\}$$

and eventually

$$\dot{z}_4 = 1 - z_2^{-1}\,\mathcal{I}m\left\{S\,e^{iz_4}\right\}.$$

Using this gives

$$\dot{z}_3 = z_1 r^{-2} - z_2^{-1}\,\mathcal{I}m\left\{S\,e^{iz_4}\right\}.$$

We have, with

$$\dot{z}_1 = \mathcal{I}m\,\bar{x}\,Q \tag{98}$$

four differential equations for z_1 through z_4.

Next, we introduce the new independent variable

$$s = \int_0^t z_1 r^{-2} dt . \tag{99}$$

For a general function f, we have

$$f' \equiv \frac{df}{ds} = \frac{df}{dt} r^2 z_1^{-1} .$$

We now write our four differential equations for z_1 through z_4 in the form

$$z' = a(z) + p(z) \tag{100}$$

where z, a, p are each column vectors with elements z_i, a_i, p_i $(i = 1, \ldots, 4)$, respectively, and $a_1(z) = a_2(z) = 0$, $a_3(z) = 1$, $a_4(z) = r^2 z_1^{-1} = t'$,

$$p_1 = a_4 \; m\bar{x}Q, \quad p_2 = -a_4 \mathrm{Re}\, P, \quad p_3 = p_4 = a_4 z_2^{-1} \mathcal{I}m\, P$$

and finally,

$$P = z_1^{-1} (e^{iz_3} - z_2) \mathcal{I}m\, \bar{x} Q - i z_1 e^{iz_4} Q . \tag{101}$$

Now Q appears only in the definition of P, and so in p_2. The vector a is independent of Q. If $Q = 0$ (i.e., $\mu = 0$), then $p = 0$, and we have the Kepler problem:

$$z' = a(z) .$$

From our existence theorem, we know a solution

$$z = \varphi(s, \zeta)$$

exists and is holomorphic in both s and ζ within some region. On inspection of a(z), we see that the solution is

$$\varphi_1 = \zeta_1, \quad \varphi_2 = \zeta_2, \quad \varphi_3 = \zeta_3 + s,$$

$$\varphi_4 = \zeta_4 + \zeta_1^3 \int_0^s (1 - \zeta_2 \cos(\zeta_3 + \sigma))^{-2} d\sigma$$

This is the solution of the unperturbed system, which is the Kepler problem, in a rotating coordinate system.

We must now write the periodicity condition in terms of x. The form is the same, since we have only a scale change. The conditions are the

$$\mathcal{I}m\, x(t_o, x_o, \dot{x}_o) = 0, \quad \mathcal{I}m\, x_o = 0$$

$$Re\, \dot{x}(t_o, x_o, \dot{x}_o) = 0, \quad Re\, \dot{x}_o = 0. \qquad (102)$$

Now, we have that

$$x_o = r_o e^{i(\zeta_3 - \zeta_4)}$$

since ζ_i is the initial value of z_i, and so $\zeta_3 - \zeta_4 = 0$, modulo π. We choose $\zeta_3 = \zeta_4 = 0$. Now

$$\dot{x} + ix = -i z_1^{-1} \left(z_2 e^{-iz_4} - e^{-i(z_3 - z_4)} \right)$$

and so, initially, we have

$$\dot{x}_o + i x_o = -i \zeta_1^{-1} (\zeta_2 - 1). \qquad (103)$$

We can now simplify the equations. We may write

$$x = \zeta_1^2 (1 - \zeta_2 \cos s)^{-1} e^{i(s - \varphi_4)}.$$

The initial conditions ζ_3 and ζ_4 have been chosen zero, so we must now choose ζ_1, ζ_2 in such a way that x describes an ellipse. Now, $\varphi_4 = t$, and so

$$x e^{it} = \zeta_1^2 (1 - \zeta_2 \cos s)^{-1} e^{is}.$$

But xe^{it} represents a vector in a rotating coordinate system. Putting $\zeta_2 = \epsilon$, we see that ζ_1 determines the major axis. We have that

$$t = \zeta_1^3 \int_0^s (1 - \epsilon \cos s)^{-2} \, ds \tag{104}$$

which we can integrate if $s = 2\pi$ to give the period, T_o say:

$$T_o = 2\pi \zeta_1^3 (1 - \epsilon^2)^{-3/2} \tag{105}$$

Now, in the rotating coordinate system, the vector xe^{it} has not necessarily returned to its initial position.

The factor e^{it} has a period 2π and the other factor, a period of T_o. We require that one differs from the other by a rational factor, i.e.,

$$2\pi m = T_o k \tag{106}$$

for some integers m and k. We call this relation a commensurability condition, or say that the period T_o is commensurable with 2π. Now, on ensuring that Equations (105) and (106) are statisfied simultaneously, the relation

$$\zeta_1^3 = \frac{m}{k} (1 - \epsilon^2)^{3/2} \tag{107}$$

must be satisfied, which we look upon as a condition upon ζ_1. The particle will make k - m circuits about the origin before it closes its orbit.

Now at time t_o, x is real, and $\dot{x}$ is purely imaginary, and t_o itself is equal to πm. The values of ζ_i $(i = 1, \ldots, 4)$ so chosen above, and satisfying Equation (107) do indeed give us a solution. We now put a * superscript on ζ and its components to show that it is initial conditions for a periodic solution when $\mu = 0$. We have ensured that the periodicity conditions are satisfied. To be able to analytically continue this solution to sufficiently small $\mu > 0$, we must have some Jacobian $J \neq 0$.

Our periodicity condition requires that if $s = s_o$ at $t = t_o$, then

$$z_3(s, \zeta_1, \zeta_2, \mu) = \pi k, \quad z_4 = \pi m$$

at $s = s_o$ and for $\zeta_3 = \zeta_4 = 0$. The problem is to satisfy these two equations simultaneously for $\mu \neq 0$. They are satisfied for $s_o^* = \pi k$.

We now compute the Jacobian of z_3, z_4 with respect to s, ζ_2 on the perpendicular solution, and evaluate at the initial conditions for $\mu = 0$. We shall find that the only non-vanishing Jacobian is

$$\left. \frac{\partial (z_2, z_4)}{\partial (s, \zeta_2)} \right|_{(s=s_o^*,\ \zeta=\zeta^*,\ \mu=0)}$$

where $\zeta^* = (\zeta_1^*, \zeta_2^*, 0, 0)$. We may assume we have a true ellipse and so $0 < \epsilon < 1$. Recall that the Equation (100) satisfies the conditions of general existence theorem for solutions to systems differential equations. From this, we deduce that solutions of Equation (100) are continuous and differentiable functions of all their variables and parameters. In particular, the solutions are holomorphic functions of μ.

Now the solution to the unperturbed system, i.e., Equation (100) with $\mu = 0$, is

$$\varphi_1 = \zeta_1, \quad \varphi_2 = \zeta_2, \quad \varphi_3 = s, \quad \varphi_4 = \zeta_1^3 \int_0^s (1 - \zeta_2 \cos\sigma)^{-2}\, d\sigma .$$

These give

$$x = \zeta_1^2 (1 - \zeta_2 \cos s)^{-1} e^{i(\varphi_4 - s)} . \tag{108}$$

Now, we want x to be periodic. As before, we let

$$r = |x| = \zeta_1^2 (1 - \zeta_2 \cos s)^{-1} ,$$

and if $\zeta_2 = 0$ and

$$\varphi_4 \,\alpha\, s$$

since $\varphi_4 = \zeta_1^3 s \; e^{i(1-\zeta_1^3)s}$, and

$$x = \zeta_1^3 e^{i(1-\zeta_1^3)s} .$$

In this case, x is always a periodic function of s. We next define $S = \frac{2\pi}{1 - \zeta_1^3}$ if $\zeta_1 \neq 1$. Note that $\zeta_1 = 1$ gives the circular solution. For the elliptic case, we require $0 < \zeta_2 < 1$.

Now consider

$$\alpha(s) \equiv s - \varphi_4 = s - \zeta_1^3 \int_0^s (1 - \zeta_2 \cos\sigma)^{-2} d\sigma . \tag{109}$$

s is increased by a multiple of 2π, and we require that $s - \varphi_4$ increases by an integral value of 2π. That $s - \varphi_4$ increases by an integral value of 2π is not so necessarily, but conditionally. We want

$$\alpha(2\pi k) = 2\pi k - \zeta_1^3 \, 2\pi k(1 - \zeta_2^2)^{-3/2} = 2\pi n \tag{110}$$

for some integers k, n. We have a condition, Equation (107), on ζ_1, ζ_2 which may be written as

$$k\zeta_1^3 (1 - \zeta_2^2)^{-3/2} = k - n = m , \tag{111}$$

and we notice, on choosing $0 < \zeta_2 < 1$ and computing ζ_1 from Equation (111), that for ζ_1, ζ_2 so chosen, Equation (110) is automatically satisfied. As s goes from 0 to $2\pi k$, r makes k revolutions about the ellipse (in fixed coordinates), and encircles the origin k - m times, to close the figure in rotating coordinate system. The figure in the rotating coordinate system is a precessing ellipse. We may assume that m is chosen positive, since we can obtain all cases on letting k take positive or negative values. The ratio m/k is the type of elliptic orbit.

One remaining point of interest is where the second orthogonal crossing of the real axis occurs. We assume k and m have no common factors, so that $t_o = \pi m$, $s_o = \pi k$. Putting these values in Equation (108) yields that the exponential term is real, and so we are again on the real axis. Further, we see that x is purely imaginary. All this has been done for $\mu = 0$.

We now want to see if the conditions we have, namely $z_3 = \pi k$, $z_4 = \pi m$ at $t = t_o$, and initially

$$\zeta_1^3 = \frac{m}{k}(1 - \zeta_2^2)^{3/2} ,$$

$\zeta_2 = \epsilon$ can be satisfied if $\mu \neq 0$. For this, we require the implicit function theorem which states:

If the equations

$$f_k(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \qquad (k = 1, \ldots, n)$$

is satisfied for the particular values

$$s^* = (x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_m^*)$$

and if the functional determinant $\partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n) \neq 0$ in some domain combining s^*, then we can find

$$x_k = \varphi_k(y_1, \ldots, y_m) \qquad (k = 1, \ldots, n)$$

such that

$$f_k(\varphi_1(y), \ldots, \varphi_n(y), y_1, \ldots, y_m) = 0 \qquad (k = 1, \ldots, n)$$

$y = (y_1, \ldots, y_m)$. Now, if we can say that $\partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n) \neq 0$ at s^* only, then we can solve for the x's as functions of y's only so long as $(y_1, \ldots, y_n)$ is sufficiently close to s^*, i.e., $\|y - y^*\|$ is sufficiently small, provided the $f_1, \ldots, f_n$ satisfy certain continuity conditions.

We now apply this with z_3, z_4 as the f's and s, ζ_2 as the x's. The y's will be ζ_1 and μ. We have no need to vary ζ_1, since we can do this with a suitable variation in ζ_2. We do want to vary μ so that $\mu \neq 0$. The implicit function theorem can be applied if $\partial (z_3, z_4)/\partial (s, \zeta_2) \neq 0$. This determinant is

$$\begin{vmatrix} 1 & 0 \\ 0 & z_{4\zeta_2} \end{vmatrix} \neq 0$$

provided $z_{4\zeta_2} \neq 0$. Now

$$z_4 = \zeta_1^3 \int_0^s (1 - \zeta_2 \cos\sigma)^{-2} \, d\sigma$$

and so

$$z_{4\zeta_2} = -\zeta_1^3 \int_0^s (1 - \zeta_2 \cos\sigma)^{-3} \cos\sigma \, d\sigma .$$

We must ensure that this condition is satisfied. If it is then, we can analytically continue our elliptical Keplerians solutions for sufficiently small $\mu > 0$.

2.6 STABILITY METHODS

We begin the discussion of this topic with a definition of stability (Reference 3) for periodic solutions of a Hamiltonian system. The restrictions to Hamiltonian systems and periodic motions is no limitation in the case of the restricted problem of three bodies with which we shall be principally concerned. To fix ideas, let us consider a Hamiltonian system

$$\dot{x}_k = E_{y_k}, \quad \dot{y}_k = -E_{x_k} \qquad (k = 1, \ldots, n) \tag{112}$$

and let

$$x = x(t, \xi, \eta), \quad y = y(t, \xi, \eta) \tag{113}$$

to be a periodic solution of Equation (112) such that $\xi = x(0, \xi, \eta)$, $\eta = y(0, \xi, \eta)$. For this solution, we set $E = \gamma$ and define R and U as follows: R is the domain of the real x-y space, of dimension 2n, on which E_{y_k}, E_{x_k} $(k = 1, \ldots, n)$ are uniformly Lipschitz continuous, and R contains our periodic solution $x = x(t)$, $y = y(t)$; and by U we denote any open subset of R which is also a neighborhood of our selected periodic solution (113). By introducing $U\gamma$, the intersection of the neighborhood U with the surface $E = \gamma$, we have a 2n-1 dimensional neighborhood of the intersection, a_γ, of our periodic solution with the hypersurface $E = \gamma$. We then speak of stability of a conservative system at a periodic solution if for each neighborhood $U\gamma$ of the given periodic solution there exists another neighborhood $V\gamma$ such that all the intersections of the trajectory through any point of $V\gamma$ and the surface $E = \gamma$ lie in $U\gamma$. We say that the solution is unstable if the only point set satisfying the above requirement is $a\gamma$ itself. If neither hold, then we say that the stability is mixed.

Now we can look upon the solutions of Equation (112) as defining a mapping, S, on points of $E = \gamma$ with the periodic solution (113) defining a fixed point of S^q, for some $q = 1, 2, \ldots, n$, n finite. From this, we may define stability of a mapping in the neighborhood in a similar manner, namely, that for every neighborhood U of a fixed point of a mapping S we require the existence of a neighborhood V such that all the images, including images of the inverse mapping, lie in U. Instability and the mixed case is defined in a similar way.

We now deduce one simple consequence of our definitions, i.e., to be stable, a mapping must have an invariant neighborhood of the fixed point. The proof of this is given by Siegel (Reference 3) as follows. Now if there exists an invariant neighborhood $V \subset U$ for every U, then the mapping is stable for all the images of points of V lie in U since $V \subset U$. We now suppose the mapping stable, and let $D \subset U$ be a neighborhood of the fixed point such that $D_n = S^n D \subset U$ for all $n = \pm 1, \pm 2, \ldots$. Then the set

$$V = \bigcup_n D_n \subset U \; (D_o = D)$$

is invariant under the mapping S. Clearly, for a continuous closed invariant curve, $\mathfrak{C}$, containing in its interior the fixed point, a, implies that the point set contained within $\mathfrak{C}$ is invariant and so the mapping is stable. The existence of such an invariant curve is, then, a sufficient condition for stability. It is upon a theorem of Moser's concerning the existence of closed invariant curves surrounding the fixed point, a, of a measure preserving mapping that we shall base our stability studies.

We shall now review the tools available to study the stability of periodic solutions of the restricted problem of three bodies. The three most important are Area Preserving Mappings, the Normal Form of the Hamiltonian and the Reduction of Perturbations.

2.6.1 Area Preserving Mappings

Within this appendix, we shall introduce the principles of area-preserving mappings as tools in the study of stability problems of periodic solutions of the restricted three-body problem. Now, we know that the solutions $x = x(t, \xi, \eta)$,

$y = y(t, \xi, \eta)$, where $\xi = x(0, \xi, \eta)$, $\eta = y(0, \xi, \eta)$, of a Hamiltonian system $x - Hy$, $y - -H_x$ define a mapping of the initial conditions, ξ, η onto the point (x, y) in phase space (Reference 3). The equations of motion of the massless particle of the restricted three-body problem may be expressed in Hamiltonian form as follows:

$$\dot{x}_k = E_{y_k}, \quad \dot{y}_k = -E_{x_k} \quad (k = 1, 2) \tag{114}$$

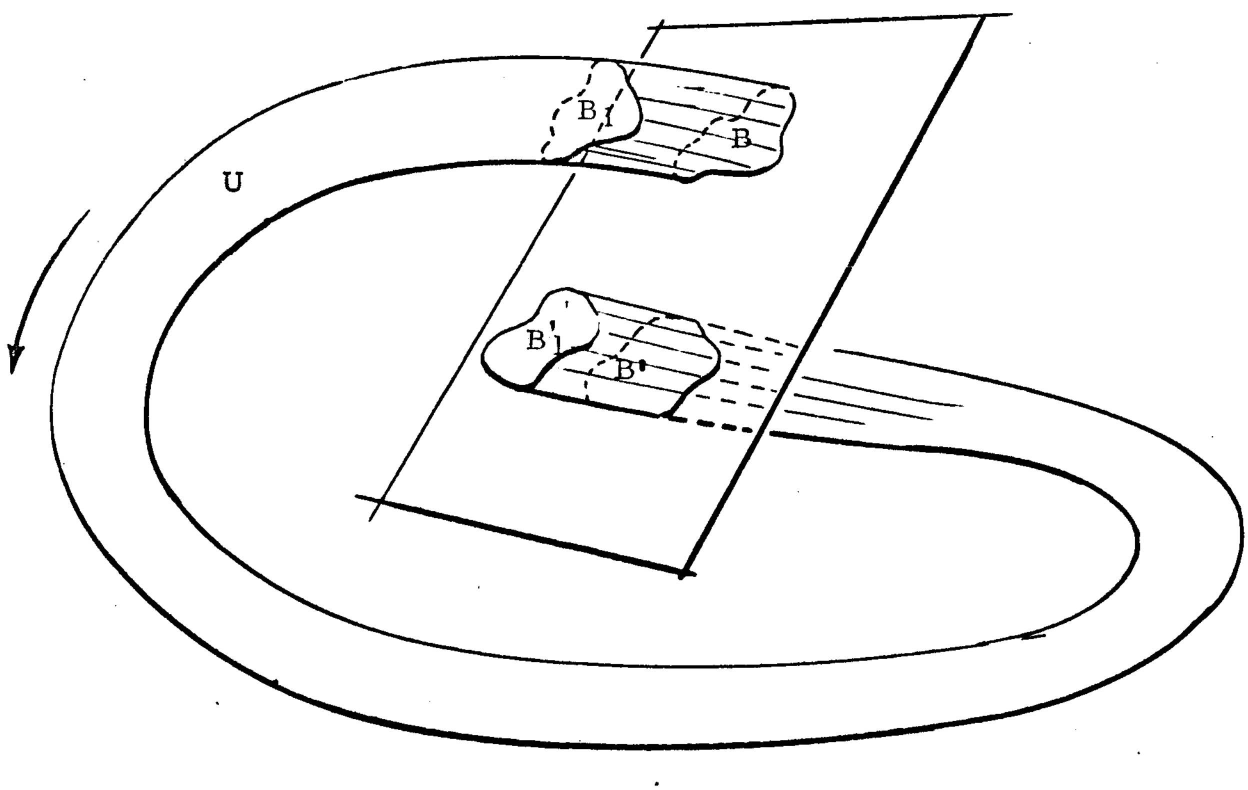
where

$$E = \tfrac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - \frac{\mu}{\left[(x_1 + \mu - 1)^2 + x_2^2\right]^{1/2}} - \frac{1 - \mu}{\left[(x_1 + \mu)^2 + x_2^2\right]^{1/2}}$$

Now, for a sensiblemotion, at least one of the E_{y_k} or $E_{x_k} \neq 0$ $(k = 1, 2)$ initially, and using this fact and the Jacobi integral, we can reduce this mapping, near a periodic solution, to a mapping, s, of a neighborhood of a two-dimensional plane in phase space into another neighborhood of the plane and s is area preserving. This result is of such fundamental importance that, perhaps, we should establish it. We require the existence of a periodic solution, and such has been shown to exist by methods of analytical continuation (References 2, 3 and 9). Now,

$$E_{x_2} = y_1 + \frac{\mu x_2}{\left[(x_1 + \mu - 1)^2 + x_2^2\right]^{3/2}} - \frac{(1 - \mu)(x_1 + \mu)}{\left[(x_1 + \mu)^2 + x_2^2\right]^{3/2}} \tag{115}$$

so that we may choose initial values $\zeta^* = (\xi_1, \xi_2, \eta_1, \eta_2 = 0)$ of (x_1, x_2, y_1, y_2) such that $E_{x_2} \neq 0$ at $t = t_o = 0$, and we must be sufficiently close to a periodic solution. Clearly, the position of the two massive bodies, $x_1 = 1 - \mu$, $x_2 = 0$ and $x_1 = \mu$, $x_2 = 0$, are singular points and must be avoided. Now, consider a neighborhood, B, of ζ^* with $\zeta_2^* = 0$, such that all solutions starting in B again pierce the surface $\eta_2 = 0$ in a set B', say. Regularity of the solutions as functions of the initial conditions ensure such a neighborhood B, and B' will be a neighborhood of the trajectory with initial conditions all those points being on the trajectories connecting the points of B to B'. We now display this diagramatically.



We now use the solutions to map this tube, which we consider as initial conditions, into points x. We choose reasonably small t with the view of letting $t \to 0$, eventually. After a time t, then, the solutions map U into U_1, where U_1 is the tube commencing at B_1 and ending in B_1'. B_1 and B_1' are not necessarily planar surfaces. Now, the mapping is area preserving, and so

$$V(U) = V(U_1) \tag{116}$$

where V(S) is the Lebesque measure of the set S. Now let R be the subset of U lieing between B and B_1, and R' be the subset of U_1, lieing between B' and B_1'. Then,

$$V(U_1) + V(R') = V(U) + V(R') \tag{117}$$

$$\therefore\ V(R) = V(R') \quad . \tag{118}$$

Now, if we have chosen B suitably, we may write Equation (118) as

$$\int_{R'} dx_1\, dx_2\, dy_1\, dy_2 = \int_{R} dx_1\, dx_2\, dy_1\, dy_2 \quad . \tag{119}$$

Now, we use the solutions,

$$x_k = x(t,\zeta), \quad y_k = y(t,\zeta) \qquad (k = 1,2;\ \eta_2 = \eta_2^*)$$

to transform to the new integration variables ξ_1, ξ_2, η_1, t.

This gives

$$\int_{R'} J\left(\frac{x_1, x_2, y_1, -y_2}{\xi_1, \xi_2, \eta_1, t}\right) d\xi_1\, d\xi_2\, d\eta_1\, dt = \int_{R} J\left(\frac{x_1, x_2, y_1, -y_2}{\xi_1, \xi_2, \eta_1, t}\right) d\xi_1\, d\xi_2\, d\eta_1\, dt .$$

Now for $t = 0$, $\dot{y}_2 = -E_{x_2} \neq 0$ and $(Z\zeta) = I$, therefore $J = -E_{x_2}$.

Substituting for J and dividing throughout by t gives

$$\int_{R} \frac{-E_{x_2}}{t}\, d\xi_1\, d\xi_2\, d\eta_1\, dt = \int_{R'} \frac{-E_{x_2}}{t}\, d\xi_1\, d\xi_2\, d\eta_1\, dt$$

Now, proceeding to the limit as $t \rightarrow 0$, we have that $R \rightarrow B$ $B' \rightarrow B'$ and so we have

$$\int_{B} -E_{x_2}\, dx_1\, dx_2\, d\xi_1 = \int_{B'} -E_{x_2}\, dx_1\, dx_2\, d\xi_1 \tag{120}$$

we have reduced the dimension of the neighborhood to 3 but have lost the measure preserving quality of our mapping. We may now restore this on using the Jacobi integral, which is just E itself. Now we choose $E = \gamma$ a constant and we wish to use this to substitute for ξ_2. The Jacobian for this transformation is just $1/E_{x_2} \neq 0$ by choice of initial conditions. Our integral then becomes

$$\int_B - \frac{E_{x_1}}{E_{x_2}} \, d\xi_1 \, d\eta_1 \, d\gamma = \int_{B'} - \frac{E_{x_2}}{E_{x_2}} \, d\xi_1 \, d\eta_1 \, d\gamma$$

and so

$$\int_B d\xi_1 \, d\eta_1 \, d\gamma = \int_{B'} d\xi_1 \, d\eta_1 \, d\gamma \quad . \tag{121}$$

But now, E is independent of time t and so, on a particular solution, E has the same value when evaluated on B or B'. So we let γ vary over the same point set in B as in B' and then B and B' are the Cartesian product of some set F and F' with γ respectively, where F and F' have points (ξ_1, η_1). We now see that

$$\int_F d\xi_1 \, d\eta_1 = \int_{F'} d\xi_1 \, d\eta_1 \tag{122}$$

and we have an area preserving mapping in the plane $\eta_2 = \eta_2$, and where $E = \gamma$ is solved for $x_2 = \xi_2$ initially.

Now we know that such a mapping has a normal form and can be classified by the determinant of the linear terms. If the mapping is hyperbolic, we know we have instability, if elliptic, then if the substitution carrying it to the normal form is convergent, then it is stable. However, the divergence of this substitution does not preclude stability. From our discussion in Section 3, we recall that the existence of closed invariant curves surrounding the fixed point are a sufficient condition for stability. Now Moser (Reference 1) has a theorem which concerns invariant curves to a perturbed stable area preserving mapping. It is this, and we shall apply it to our problems.

Let there be given an area preserving mapping, M_o:

$$\begin{aligned} \theta_1 &= \theta + \alpha(r) \\ r_1 &= r \end{aligned} \tag{123}$$

which we recognize as the normal form of an area preserving mapping in plane polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. We call this a twist mapping. Let

$$\begin{aligned} \theta_1 &= \theta + \alpha(r) + f(\theta, r) \\ r &= r + g(\theta, r) \end{aligned} \tag{124}$$

be a perturbation of this mapping then under certain conditions this mapping has invariant curves near concentric circles. So far as we are immediately concerned, his application to the restricted problem of three bodies is of prime importance.

In References 3 and 4, the mapping near a fixed point is shown to be, in complex form,

$$Z_1 = Z\, e^{i\alpha} + F(Z, \bar{Z}) = f(Z, \bar{Z}) \tag{125}$$

where F vanishes at least quadratically at the fixed point $Z = 0$, and $Z = x + iy$. It is area preserving if

$$\frac{\partial(f, \bar{f})}{\partial(Z, \bar{Z})} = 1$$

Such a mapping may be transformed to

$$w_1 = w\, e^{i(\alpha + \beta |w|^s)} + O_\ell(|w|^q) \tag{126}$$

where s is an even integer, $o < s < q-1$ and of importance later $\beta = 0, +1$ or -1, and β is independent of the choice of $q > s + 1$. Ignoring the error term $O_\ell(|w|^{s+2})$, one has a twist mapping. Moser, then shows that his theorem is applicable and even further that the mapping is stable for $\beta \neq 0$. The number α must be such that

$$\frac{v\alpha}{2\pi} \neq \text{integer} \qquad v = 1, 2, \ldots, s + 2.$$

We need only compute α, β approximately. The application to the three-body problem, with which he follows the main result, is of particular interest to us. He restricts his considerations to "sufficiently small $\mu > 0$." We are interested in finite values of μ, in particular $\mu = 1/80$ for the earth-moon system. The problem then is to establish the mappings in the form (126) for given values of μ or ranges of μ and then discuss the stability of the system.

In this approach, the method of Moser (Reference 2) may be applied directly to certain periodic solutions, e.g., first kind of Poincaré. To do this, we must write the mapping defined by the solutions of the restricted three-body problem in the form (References 1 and 3),

$$w_1 = w\, e^{i(\alpha+\beta|w|^s)} + O_\ell\,(|w|^q) \tag{127}$$

where s is an even integer, $0 < s < q-1$, α and β are numbers and $\beta = 0$, $+1$ or -1, and is independent of the choice of $q > s + 1$. The notation $O_\ell(|w|\, q)$ means that for some $c > 1$, the inequalities

$$|w|^{\rho+\sigma} \left| \frac{\partial^{\rho+\sigma}}{\partial U^\rho\, \partial V^\sigma} \right| G \;\leq\; c\, |w|^s$$

hold for $|w| < c^{-1}$ and $\rho + \sigma \leq \ell$ where G is the perturbation of our twist mapping (127). Now Moser (Reference 1) shows us that the mapping is stable for $\beta \neq 0$. We must compute α, β for the selected mappings and further, these values need only be computed approximately for the only values of β possible are 0, -1 or -1.

2.6.2 The Normal Form of the Hamiltonian

The second method useful in the study of Hamiltonian systems is the normalization of the Hamiltonian. The most recent development is due to Arnol'd (Reference 5) and is of particular interest for equilibrium solutions, e.g., the libration points L_4, L_5 of the restricted three-body problem. Birkhoff (Reference 6) has shown that the Hamiltonian can be written in the form

$$K = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \alpha x_1^2 y_1^2 + \beta x_1 y_1 x_2 y_2 + \gamma x_2^2 y_2^2 + \ldots \qquad (128)$$

Now Arnold's theorem states that if

1. λ_1 and λ_2 are purely imaginary
2. $\lambda_1/\lambda_2 \notin m$ where m is some set of measure zero on the real axis.
3. $\Phi \equiv \alpha\lambda_2^2 + \beta\lambda_1\lambda_2 + \gamma\lambda_1^2 \neq 0$

then the equilibrium solution $x_1 = y_1 = x_2 = y_2 = 0$ is stable. This theorem has been applied to the equilateral libration points of the restricted three-body problem by Leontovič (Reference 7). This method does not appear to have a direct application to the problems we wish to study although there may be an application to the study of the rate of growth of the divergence of motions near L_1.

2.6.3 The Reduction of Perturbations

Tasks 3 and 4 are intended to find the deviation of perturbed motions from the nominal after a finite time. One method is to use a digital computer and generate these motions numerically, but the computed behavior is not proof, only an indication of the nature of the motion. However, it may be possible to use Arenstorf's Reduction of Perturbation method (Reference 8).

To use this method, we must be able to write our system of differential equations as a perturbation of a system with a known solution. A time-dependent coordinate transformation is then performed iteratively to reduce the perturbed system to the basic system. It is thought that this method may be used to compute the decay of certain periodic solutions of the three-body problem under the perturbing influence of a fourth body. A computer program and plot output would effectively display the nature of the decay of such an orbit.

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